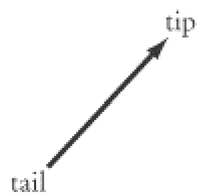
Vectors

VECTORS ARE USUALLY THE FIRST THING you learn in a physics class, and they're the first thing you'll learn here. Vectors are one of the fundamental mathematical tools the physicist uses, and one that is frequently misunderstood or misapplied by students. Generally, there aren't more than one or two questions on SAT II Physics that test your knowledge of vectors directly, but there are a host of problems—particularly in mechanics—where arriving at the right solution demands a solid grasp of how to apply and manipulate vectors. Even if you feel confident with vectors, we urge you to review this chapter and be absolutely sure you won't get tripped up on what would otherwise be some easy questions.

What's a Vector?

A **vector** is a mathematical object possessing, and fully described by, a **magnitude** and a **direction**. It's possible to talk about vectors simply in terms of numbers, but it's often a lot easier to represent them graphically as arrows. The vector's magnitude is equal to the length of the arrow, and its direction corresponds to where the arrow is pointing. Physicists commonly refer to the point of a vector as its **tip** and the base as its **tail**.



There are a number of ways to label vectors. You may have seen vectors labeled $\vec{A}$ or $\mathbf{A}$. This book will follow the convention you'll find on SAT II Physics: vectors are written in boldface and vector magnitudes in plain script. For example, vector $\mathbf{A}$ has magnitude A.

Vectors vs. Scalars

In contrast to a vector quantity, a **scalar** quantity does not have a direction; it is fully described by just a magnitude. Examples of scalar quantities include the number of words in this sentence and the mass of the Hubble Space Telescope. Vector quantities you'll likely come across quite frequently in physics include displacement, $\mathbf{s}$; velocity, $\mathbf{v}$; acceleration, $\mathbf{a}$; force, $\mathbf{F}$; momentum, $\mathbf{p}$; electric field, $\mathbf{E}$; and magnetic field, $\mathbf{B}$.

When in doubt, ask yourself if a certain quantity comes with a direction. If it does, it's a vector. If it doesn't, it's a scalar.

EXAMPLE

Which of the following sentences deal with vector quantities?

I. "I used to drive a 10-ton truck."
II. "You'll find a gas station if you follow this road 20 miles due north."
III. "The 10-volt battery is the one on your left."

(A) I only

(B) II only

(C) III only

(D) II and III
(E) I, II, and III

"I used to drive a 10-ton truck" deals with mass, which is a scalar quantity. When we know that a truck weighs 10 tons, we don't need to ask, "in what direction?" "You'll find a gas station if you follow this road 20 miles due north" deals with the vector quantity of displacement. When asking directions to a gas station, you don't simply want to know how far it is from where you are, but also in what direction you need to go. "The 10-volt battery is the one on your left" is slightly tricky: volts are a scalar quantity—you don't ask in what direction the battery's volts are going. However, you might be deceived by the mention of "on your left." However, "on your left" is a reference to the battery, not to the volts. The magnitude "10 volts" doesn't have a direction, so that quantity is a scalar. The answer is **B**.

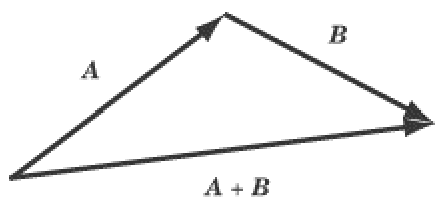
Vector Addition

There are bound to be several questions on SAT II Physics that involve vector addition, particularly in mechanics. The test doesn't demand a very sophisticated understanding of vector addition, but it's important that you grasp the principle. That is, you won't be asked to make complicated calculations, but you will be expected to know what happens when you add two vectors together.

The easiest way to learn how vector addition works is to look at it graphically. There are two equivalent ways to add vectors graphically: the **tip-to-tail method** and the **parallelogram method**. Both will get you to the same result, but one or the other is more convenient depending on the circumstances.

Tip-to-Tail Method

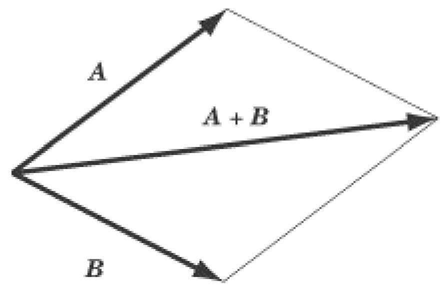
We can add any two vectors, $\boldsymbol{A}$ and $\boldsymbol{B}$, by placing the tail of $\boldsymbol{B}$ so that it meets the tip of $\boldsymbol{A}$. The sum, $\boldsymbol{A} + \boldsymbol{B}$, is the vector from the tail of $\boldsymbol{A}$ to the tip of $\boldsymbol{B}$.



Note that you'll get the same vector if you place the tip of $\boldsymbol{B}$ against the tail of $\boldsymbol{A}$. In other words, $\boldsymbol{A} + \boldsymbol{B}$ and $\boldsymbol{B} + \boldsymbol{A}$ are equivalent.

Parallelogram Method

To add $\boldsymbol{A}$ and $\boldsymbol{B}$ using the parallelogram method, place the tail of $\boldsymbol{B}$ so that it meets the tail of $\boldsymbol{A}$. Take these two vectors to be the first two adjacent sides of a parallelogram, and draw in the remaining two sides. The vector sum, $\boldsymbol{A} + \boldsymbol{B}$, extends from the tails of $\boldsymbol{A}$ and $\boldsymbol{B}$ across the diagonal to the opposite corner of the parallelogram. If the vectors are perpendicular and unequal in magnitude, the parallelogram will be a rectangle. If the vectors are perpendicular and equal in magnitude, the parallelogram will be a square.

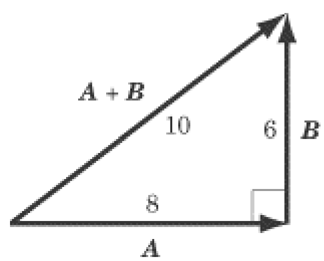


Adding Vector Magnitudes

Of course, knowing what the sum of two vectors looks like is often not enough. Sometimes you'll need to know the magnitude of the resultant vector. This, of course, depends not only on the magnitude of the two vectors you're adding, but also on the angle between the two vectors.

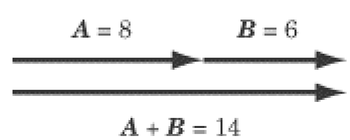
Adding Perpendicular Vectors

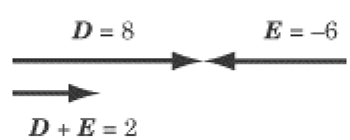
Suppose vector ***A*** has a magnitude of 8, and vector ***B*** is perpendicular to ***A*** with a magnitude of 6. What is the magnitude of ***A*** + ***B***? Since vectors ***A*** and ***B*** are perpendicular, the triangle formed by ***A***, ***B***, and ***A*** + ***B*** is a right triangle. We can use the Pythagorean Theorem to calculate the magnitude of ***A*** + ***B***, which is $\sqrt{8^2 + 6^2} = \sqrt{100} = 10.$



Adding Parallel Vectors

If the vectors you want to add are in the same direction, they can be added using simple arithmetic. For example, if you get in your car and drive eight miles east, stop for a break, and then drive six miles east, you will be $8 + 6 = 14$ miles east of your origin. If you drive eight miles east and then six miles west, you will end up $8 - 6 = 2$ miles east of your origin.





Adding Vectors at Other Angles

When **A** and **B** are neither perpendicular nor parallel, it is more difficult to calculate the magnitude of **A** + **B** because we can no longer use the Pythagorean Theorem. It is possible to calculate this sum using trigonometry, but SAT II Physics will never ask you to do this. For the most part, SAT II Physics will want you to show graphically what the sum will look like, following the tip-to-tail or parallelogram methods. On the rare occasions that you need to calculate the sum of vectors that are not perpendicular, you will be able to use the component method of vector addition, explained later in this chapter.

EXAMPLE

> Vector **A** has a magnitude of 9 and points due north, vector **B** has a magnitude of 3 and points due north, and vector **C** has a magnitude of 5 and points due west. What is the magnitude of the resultant vector, **A** + **B** + **C**?

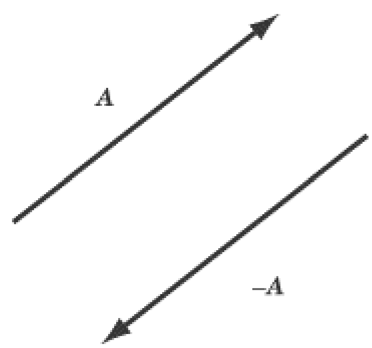
First, add the two parallel vectors, **A** and **B**. Because they are parallel, this is a simple matter of straightforward addition: 9 + 3 = 12. So the vector **A** + **B** has a magnitude of 12 and points due north. Next, add **A** + **B** to **C**. These two vectors are perpendicular, so apply the Pythagorean Theorem:

$$\sqrt{12^2 + 5^2} = 13$$

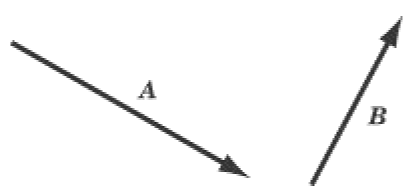
The sum of the three vectors has a magnitude of 13. Though a little more time-consuming, adding three vectors is just as simple as adding two.

Vector Subtraction

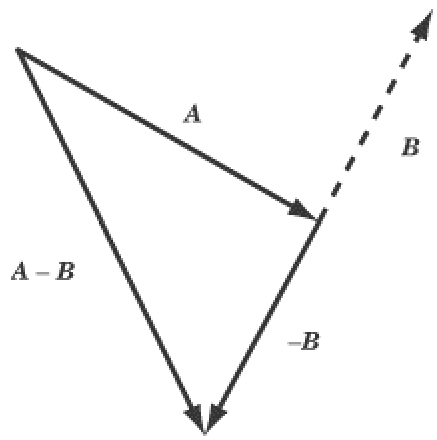
---You probably know that subtraction is the same thing as adding a negative: 8 – 5 is the same thing as 8 + (–5). The easiest way to think about vector subtraction is in terms of adding a negative vector. What's a negative vector? It's the same vector as its positive counterpart, only pointing in the opposite direction.



A – **B**, then, is the same thing as **A** + (–**B**). For instance, let's take the two vectors **A** and **B**:

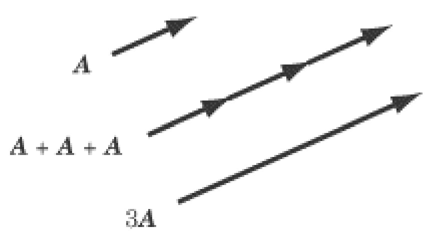


To subtract ***B*** from ***A***, take a vector of the same magnitude as ***B***, but pointing in the opposite direction, and add that vector to ***A***, using either the tip-to-tail method or the parallelogram method.



Multiplication by a Scalar

Multiplication is like repeated addition. Multiplying 4 by 3 means adding four three times: $3 \times 4 = 4 + 4 + 4 = 12$. The multiplication of a vector times a scalar works in the same way. Multiplying the vector ***A*** by the positive scalar *c* is equivalent to adding together *c* copies of the vector ***A***. Thus $3\mathbf{A} = \mathbf{A} + \mathbf{A} + \mathbf{A}$. Multiplying a vector by a scalar will get you a vector with the same direction, but different magnitude, as the original.



The result of multiplying ***A*** by *c* is a vector in the same direction as ***A***, with a magnitude of $c \times A$. If *c* is negative, then the direction of ***A*** is reversed by scalar multiplication.

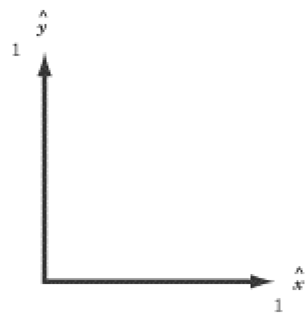
Vector Components

As we have seen, vector addition and scalar multiplication can produce new vectors out of old ones. For instance, we produce the vector $\mathbf{A} + \mathbf{B}$ by adding the two vectors ***A*** and ***B***. Of course, there is nothing that makes $\mathbf{A} + \mathbf{B}$ at all distinct as a vector from ***A*** or ***B***: all three have magnitudes and directions. And just as $\mathbf{A} + \mathbf{B}$ can be construed as the sum of two other vectors, so can ***A*** and ***B***. In problems involving vector addition, it’s often convenient to break a vector down into two **components**, that is, two vectors whose sum is the vector in question.

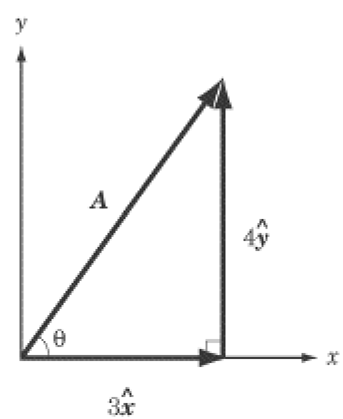
Basis Vectors

We often graph vectors in a xy-coordinate system, where we can talk about vectors in purely numerical terms. For instance, the vector $(3,4)$ is the vector whose tail is at the origin and whose tip is at the point $(3,4)$ on the coordinate plane. From this coordinate, you can use the Pythagorean Theorem to calculate that the vector's magnitude is 5 and trigonometry to calculate that its direction is about $53.1°$ above the x-axis.

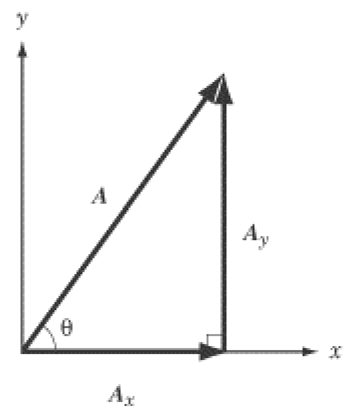
Two vectors of particular note are (1,0), the vector of magnitude 1 that points along the x-axis, and $(0,1)$, the vector of magnitude 1 that points along the y-axis. These are called the **basis vectors** and are written with the special hat notation: $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$ respectively.



The basis vectors are important because you can express any vector in terms of the sum of multiples of the two basis vectors. For instance, the vector $(3,4)$ that we discussed above—call it $\boldsymbol{A}$—can be expressed as the vector sum $3\hat{\boldsymbol{x}} + 4\hat{\boldsymbol{y}}$.



The vector $3\hat{\boldsymbol{x}}$ is called the "x-component" of $\boldsymbol{A}$ and the $4\hat{\boldsymbol{y}}$ is called the "y-component" of $\boldsymbol{A}$. In this book, we will use subscripts to denote vector components. For example, the x-component of $\boldsymbol{A}$ is $\boldsymbol{A}_x$ and the y-component of vector $\boldsymbol{A}$ is $\boldsymbol{A}_y$.



The direction of a vector can be expressed in terms of the angle θ by which it is rotated counterclockwise from the *x*-axis.

Vector Decomposition

The process of finding a vector's components is known as "resolving," "decomposing," or "breaking down" a vector. Let's take the example, illustrated above, of a vector, $\boldsymbol{A}$, with a magnitude of A and a direction θ above the *x*-axis. Because A_x, A_y, and $\boldsymbol{A}$ form a right triangle, we can use trigonometry to solve this problem. Applying the trigonometric definitions of cosine and sine,

$$\cos\theta = \frac{\boldsymbol{A}_x}{\boldsymbol{A}}$$
$$\sin\theta = \frac{\boldsymbol{A}_y}{\boldsymbol{A}}$$

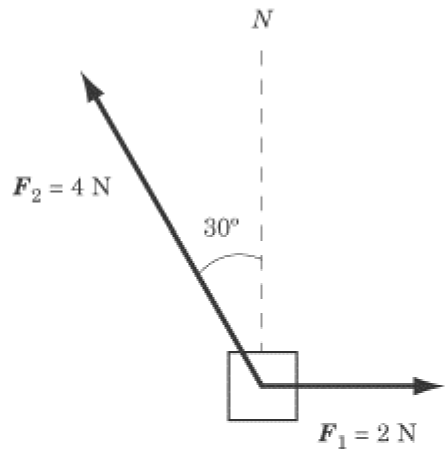
we find:

$$\boldsymbol{A}_x = A\cos\theta\hat{\boldsymbol{x}}$$
$$\boldsymbol{A}_y = A\sin\theta\hat{\boldsymbol{y}}$$

Vector Addition Using Components

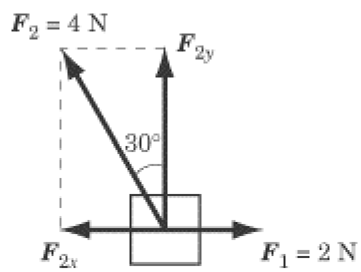
Vector decomposition is particularly useful when you're called upon to add two vectors that are neither parallel nor perpendicular. In such a case, you will want to resolve one vector into components that run parallel and perpendicular to the other vector.

EXAMPLE



Two ropes are tied to a box on a frictionless surface. One rope pulls due east with a force of 2.0N. The second rope pulls with a force of 4.0N at an angle 30Â° west of north, as shown in the diagram. What is the total force acting on the box?

To solve this problem, we need to resolve the force on the second rope into its northward and westward components.



Because the force is directed 30° west of north, its northward component is

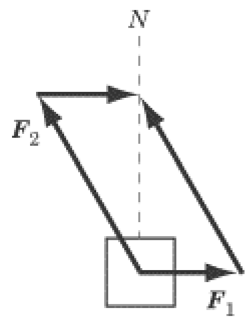
$$4.0 \cos 30° \approx 4.0 \times 0.86 = 3.4$$

and its westward component is

$$4.0 \sin 30° = 4.0 \times \tfrac{1}{2} = 2.0$$

Since the eastward component is also 2.0N, the eastward and westward components cancel one another out. The resultant force is directed due north, with a force of approximately 3.4N.

You can justify this answer by using the parallelogram method. If you fill out the half-completed parallelogram formed by the two vectors in the diagram above, you will find that the opposite corner of the parallelogram is directly above the corner made by the tails of those two vectors.



Vector Multiplication

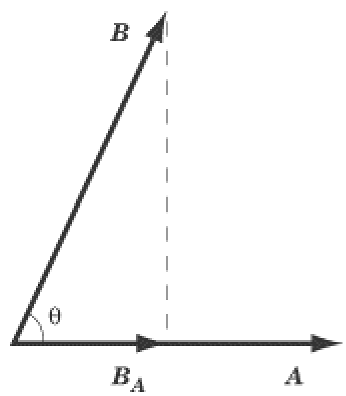
There are two forms of vector multiplication: one results in a scalar, and one results in a vector.

Dot Product

The **dot product**, also called the scalar product, takes two vectors, "multiplies" them together, and produces a scalar. The smaller the angle between the two vectors, the greater their dot product will be. A common example of the dot product in action is the formula for work, which you will encounter in Chapter 4. Work is a scalar quantity, but it is measured by the magnitude of force and displacement, both vector quantities, and the degree to which the force and displacement are parallel to one another.
The dot product of any two vectors, $\mathbf{A}$ and $\mathbf{B}$, is expressed by the equation:

$$\mathbf{A} \cdot \mathbf{B} = AB\cos\theta$$

where θ is the angle made by A and B when they are placed tail to tail.



The dot product of $\mathbf{A}$ and $\mathbf{B}$ is the value you would get by multiplying the magnitude of $\mathbf{A}$ by the magnitude of the component of $\mathbf{B}$ that runs parallel to $\mathbf{A}$. Looking at the figure above, you can get $\mathbf{A} \cdot \mathbf{B}$ by multiplying the magnitude of $\mathbf{A}$ by the magnitude of $\mathbf{B}_A$, which equals $\mathbf{B}\cos\theta$. You would get the same result if you multiplied the magnitude of $\mathbf{B}$ by the magnitude of $\mathbf{A}_B$, which equals $\mathbf{A}\cos\theta$.

Note that the dot product of two identical vectors is their magnitude squared, and that the dot product of two perpendicular vectors is zero.

EXAMPLE

> Suppose the hands on a clock are vectors, where the hour hand has a length of 2 and the minute hand has a length of 4. What is the dot product of these two vectors when the clock reads 2 o'clock?

The angle between the hour hand and the minute hand at 2 o'clock is 60°. With this information, we can simply plug the numbers we have into the formula for the dot product:

$$\text{minute hand} \cdot \text{hour hand} = (2)(4)\cos 60° = 4$$

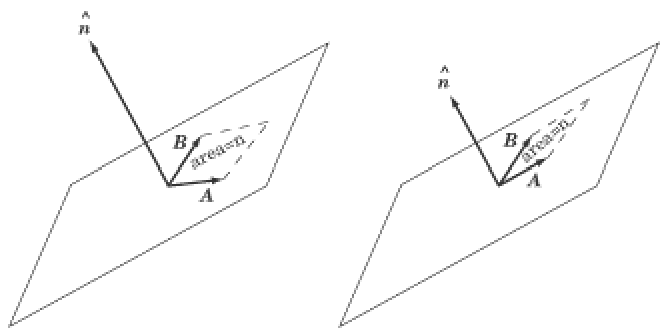
The Cross Product

The **cross product**, also called the vector product, “multiplies” two vectors together to produce a third vector, which is perpendicular to both of the original vectors. The closer the angle between the two vectors is to the perpendicular, the greater the cross product will be. We encounter the cross product a great deal in our discussions of magnetic fields. Magnetic force acts perpendicular both to the magnetic field that produces the force, and to the charged particles experiencing the force.

The cross product can be a bit tricky, because you have to think in three dimensions. The cross product of two vectors, $\boldsymbol{A}$ and $\boldsymbol{B}$, is defined by the equation:

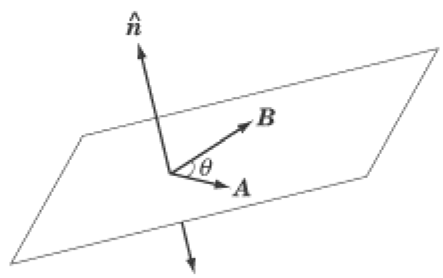
$$\boldsymbol{A} \times \boldsymbol{B} = AB\sin\theta\hat{\boldsymbol{n}}$$

where $\hat{\boldsymbol{n}}$ is a unit vector perpendicular to both $\boldsymbol{A}$ and $\boldsymbol{B}$. The magnitude of the cross product vector is equal to the area made by a parallelogram of $\boldsymbol{A}$ and $\boldsymbol{B}$. In other words, the greater the area of the parallelogram, the longer the cross product vector.

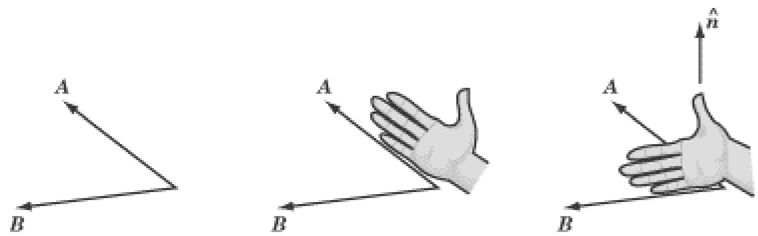


The Right-Hand Rule

You may have noticed an ambiguity here. The two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ always lie on a common plane and there are two directions perpendicular to this plane: “up” and “down.”



There is no real reason why we should choose the “up” or the “down” direction as the right one, but it’s important that we remain consistent. To that end, everybody follows the convention known as the **right-hand rule**. In order to find the cross product, $\boldsymbol{A} \times \boldsymbol{B}$: Place the two vectors so their tails are at the same point. Align your right hand along the first vector, $\boldsymbol{A}$, such that the base of your palm is at the tail of the vector, and your fingertips are pointing toward the tip. Then curl your fingers via the small angle toward the second vector, $\boldsymbol{B}$. If $\boldsymbol{B}$ is in a clockwise direction from $\boldsymbol{A}$, you’ll find you have to flip your hand over to make this work. The direction in which your thumb is pointing is the direction of $\hat{\boldsymbol{n}}$, and the direction of $\boldsymbol{A} \times \boldsymbol{B}$.



Note that you curl your fingers from $\mathbf{A}$ to $\mathbf{B}$ because the cross product is $\mathbf{A} \times \mathbf{B}$. If it were written $\mathbf{B} \times \mathbf{A}$, you would have to curl your fingers from $\mathbf{B}$ to $\mathbf{A}$, and your thumb would point downward. The order in which you write the two terms of a cross product matters a great deal. If you are right-handed, be careful! While you are working hard on SAT II Physics, you may be tempted to use your left hand instead of your right hand to calculate a cross product. Don't do this.

EXAMPLE

> Suppose once again that the minute hand of a clock is a vector of magnitude 4 and the hour hand is a vector of magnitude 2. If, at 5 o'clock, one were to take the cross product of the minute hand $\times$ the hour hand, what would the resultant vector be?

First of all, let's calculate the magnitude of the cross product vector. The angle between the hour hand and the minute hand is 150°:

$$\text{minute hand} \times \text{hour hand} = (4)(2)\sin 150°\mathbf{n} = 4\mathbf{n}$$

Using the right-hand rule, you'll find that, by curling the fingers of your right hand from 12 o'clock toward 5 o'clock, your thumb points in toward the clock. So the resultant vector has a magnitude of 4 and points into the clock.

Key Formulas

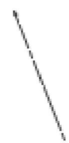
Dot Product	$\mathbf{A} \cdot \mathbf{B} = AB\cos\theta$
Cross Product	$\mathbf{A} \times \mathbf{B} = AB\sin\theta\hat{\mathbf{n}}$
Magnitude	$A = \sqrt{A_x^2 + A_y^2}$
Direction	$\theta = \arctan\frac{\mathbf{A}_y}{\mathbf{A}_x}$
***X*-, *Y*-Components**	$\mathbf{A}_x = A\cos\theta\hat{\mathbf{x}}$ $\mathbf{A}_y = A\sin\theta\hat{\mathbf{y}}$
Vector Addition	$\mathbf{A} + \mathbf{B} = (A_x + B_x)\hat{\mathbf{x}} + (A_y + B_y)\hat{\mathbf{y}}$

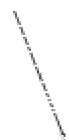
Practice Questions

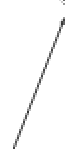
1. Which of the following vectors best represents the vector ***A*** +***B***?

   (A) 

   (B) 

   (C) 

   (D) 

   (E) 

2. Vector ***A*** has a magnitude of 5 in the leftward direction and ***B*** has a magnitude of 2 in the rightward direction. What is the value of 2***A*** – ***B***?

   (A) 12 in the leftward direction

   (B) 10 in the leftward direction

   (C) 8 in the leftward direction

   (D) 8 in the rightward direction

   (E) 12 in the rightward direction

3. When the tail of vector ***A*** is set at the origin of the *xy*-axis, the tip of ***A*** reaches (3,6). When the tail of vector ***B*** is set at the origin of the *xy*-axis, the tip of ***B*** reaches (–1,5). If the tail of vector ***A*** – ***B*** were set at the origin of the *xy*-axis, what point would its tip touch?

   (A) (2,11)

   (B) (2,1)

   (C) (–2,7)

   (D) (4,1)

   (E) (4,11)

4. ***A*** and ***B*** are vectors, and θ is the angle between them. What can you do to maximize ***A*** Â·***B***?

   I. Maximize the magnitude of ***A***
   II. Maximize the magnitude of ***B***
   III. Set θ to 90Â°

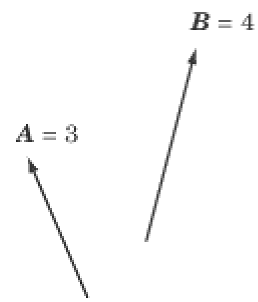
(A) None of the above

(B) I only

(C) III only

(D) I and II only

(E) I, II, and III



5. Which of the following statements is NOT true about $\boldsymbol{A} \times \boldsymbol{B}$?

(A) It is a vector that points into the page

(B) It has a magnitude that is less than or equal to 12

(C) It has no component in the plane of the page

(D) The angle it makes with ***B*** is less than the angle it makes with ***A***

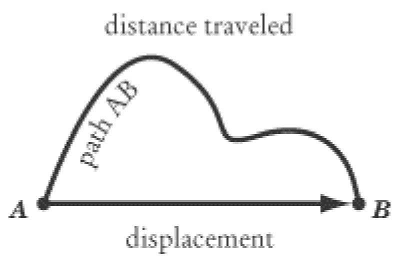
(E) It is the same as $-B \times A$

Kinematics

KINEMATICS DERIVES ITS NAME FROM the Greek word for "motion," *kinema*. Before we can make any headway in physics, we have to be able to describe how bodies move. Kinematics provides us with the language and the mathematical tools to describe motion, whether the motion of a charging pachyderm or a charged particle. As such, it provides a foundation that will help us in all areas of physics. Kinematics is most intimately connected with dynamics: while kinematics describes motion, dynamics explains the causes for this motion.

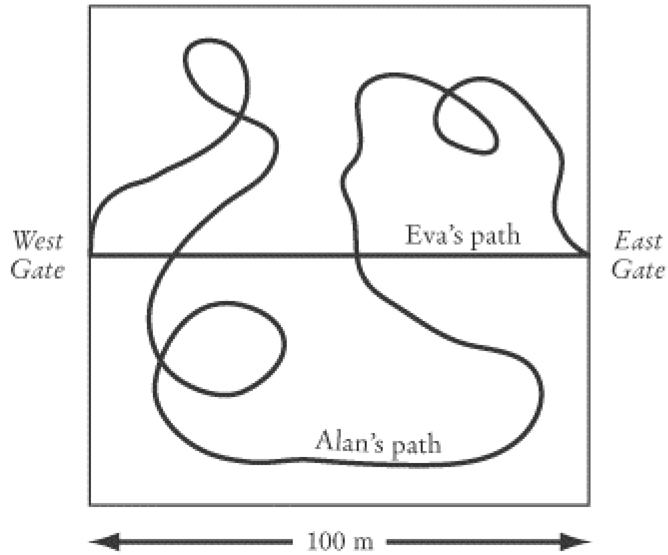
Displacement

Displacement is a vector quantity, commonly denoted by the vector $\mathbf{s}$, that reflects an object's change in spatial position. The displacement of an object that moves from point A to point B is a vector whose tail is at A and whose tip is at B. Displacement deals only with the separation between points A and B, and not with the path the object followed between points A and B. By contrast, the **distance** that the object travels is equal to the length of path AB.



Students often mistake displacement for distance, and SAT II Physics may well call for you to distinguish between the two. A question favored by test makers everywhere is to ask the displacement of an athlete who has run a lap on a 400-meter track. The answer, of course, is zero: after running a lap, the athlete is back where he or she started. The distance traveled by the athlete, and not the displacement, is 400meters.

EXAMPLE

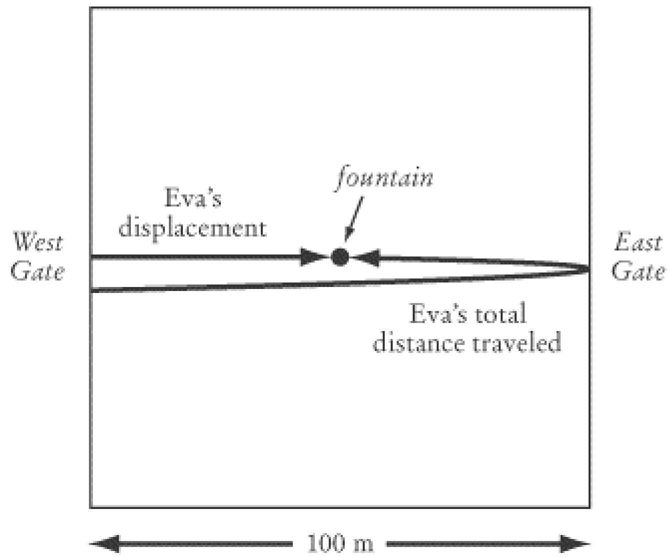


> Alan and Eva are walking through a beautiful garden. Because Eva is very worried about the upcoming SAT II Physics Test, she takes no time to smell the flowers and instead walks on a straight path from the west garden gate to the east gate, a distance of 100 meters. Alan, unconcerned about the test, meanders off the straight path to smell all the flowers in sight. When Alan and Eva meet at the east gate, who has walked a greater distance? What are their displacements?

Since Eva took the direct path between the west and east garden gates and Alan took an indirect path, Alan has traveled a much greater distance than Eva. Yet, as we have discussed, displacement is a vector quantity that measures the distance separating the starting point from the ending point: the path taken between the two points is irrelevant. So Alan and Eva both have the same displacement: 100 meters east of the west gate. Note that, because displacement is a vector quantity, it is not enough to say that the displacement is 100 meters: you must also state the direction of that displacement. The distance that Eva has traveled is exactly equal to the magnitude of her displacement: 100 meters.

> After reaching the east gate, Eva and Alan notice that the gate is locked, so they must turn around and exit the garden through the west gate. On the return trip, Alan again wanders off to smell the flowers, and Eva travels the path directly between the gates. At the center of the garden, Eva stops to throw a penny into a fountain. At this point, what is her displacement from her starting point at the west gate?

Eva is now 50 meters from the west gate, so her displacement is 50 meters, even though she has traveled a total distance of 150 meters.



When Alan and Eva reconvene at the west gate, their displacements are both zero, as they both began and ended their garden journey at the west gate. The moral of the story? Always take time to smell the flowers!

Speed, Velocity, and Acceleration

Along with displacement, **velocity** and **acceleration** round out the holy trinity of kinematics. As you'll see, all three are closely related to one another, and together they offer a pretty complete understanding of motion. **Speed**, like distance, is a scalar quantity that won't come up too often on SAT II Physics, but it might trip you up if you don't know how to distinguish it from velocity.

Speed and Velocity

As distance is to displacement, so speed is to velocity: the crucial difference between the two is that speed is a scalar and velocity is a vector quantity. In everyday conversation, we usually say speed when we talk about how fast something is moving. However, in physics, it is often important to determine the direction of this motion, so you'll find velocity come up in physics problems far more frequently than speed.

A common example of speed is the number given by the speedometer in a car. A speedometer tells us the car's speed, not its velocity, because it gives only a number and not a direction. Speed is a measure of the distance an object travels in a given length of time:

$$\text{average speed} = \frac{\text{distance traveled}}{\text{time elapsed}} = \frac{\Delta x}{\Delta t}$$

Velocity is a vector quantity defined as rate of change of the displacement vector over time:

$$\text{average velocity} = \frac{\text{change in displacement}}{\text{time elapsed}} = \frac{\Delta s}{\Delta t}$$

It is important to remember that the average speed and the magnitude of the average velocity may not be equivalent.

Instantaneous Speed and Velocity

The two equations given above for speed and velocity discuss only the *average* speed and *average* velocity over a given time interval. Most often, as with a car's speedometer, we are not interested in an average speed or velocity, but in the **instantaneous velocity** or speed at a given moment. That is, we don't want to know how many meters an object covered in the past ten seconds; we want to know how fast that object is moving *right now*. Instantaneous velocity is not a tricky concept: we simply take the equation above and assume that Δt is very, very small.

Most problems on SAT II Physics ask about an object's instantaneous velocity rather than its average velocity or speed over a given time frame. Unless a question specifically asks you about the average velocity or speed over a given time interval, you can safely assume that it is asking about the instantaneous velocity at a given moment.

EXAMPLE

Which of the follow sentences contains an example of instantaneous velocity?

(A) "The car covered 500 kilometers in the first 10 hours of its northward journey."

(B) "Five seconds into the launch, the rocket was shooting upward at 5000 meters per second."

(C) "The cheetah can run at 70 miles per hour."

(D) "Moving at five kilometers per hour, it will take us eight hours to get to the base camp."

(E) "Roger Bannister was the first person to run one mile in less than four minutes."

Instantaneous velocity has a magnitude and a direction, and deals with the velocity at a particular instant in time. All three of these requirements are met only in **B**. **A** is an example of average velocity, **C** is an example of instantaneous speed, and both **D** and **E** are examples of average speed.

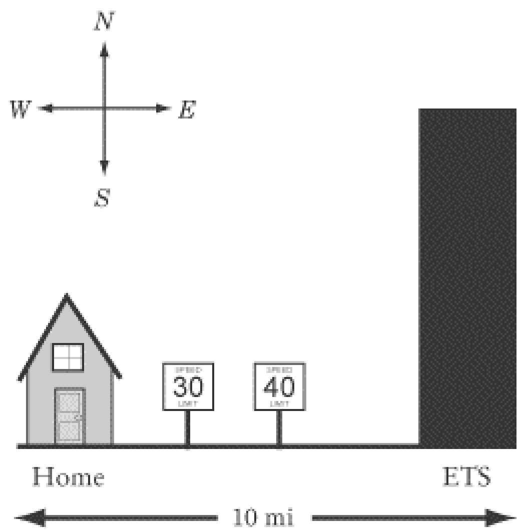
Acceleration

Speed and velocity only deal with movement at a constant rate. When we speed up, slow down, or change direction, we want to know our **acceleration**. Acceleration is a vector quantity that measures the rate of change of the velocity vector with time:

$$\text{average acceleration} = \frac{\text{change in velocity}}{\text{time elapsed}} = \frac{\Delta v}{\Delta t}$$

Applying the Concepts of Speed, Velocity, and Acceleration

With these three definitions under our belt, let's apply them to a little story of a zealous high school student called Andrea. Andrea is due to take SAT II Physics at the ETS building 10 miles due east from her home. Because she is particularly concerned with sleeping as much as possible before the test, she practices the drive the day before so she knows exactly how long it will take and how early she must get up.



Instantaneous Velocity

After starting her car, she zeros her odometer so that she can record the exact distance to the test center. Throughout the drive, Andrea is cautious of her speed, which is measured by her speedometer. At first she is careful to drive at exactly 30 miles per hour, as advised by the signs along the road. Chuckling to herself, she notes that her instantaneous velocity—a vector quantity—is 30 miles per hour due east.

Average Acceleration

Along the way, Andrea sees a new speed limit sign of 40 miles per hour, so she accelerates. Noting with her trusty wristwatch that it takes her two seconds to change from 30 miles per hour due east to 40 miles per hour due east, Andrea calculates her average acceleration during this time frame:

$$\begin{aligned}\text{average acceleration} &= \frac{40\text{ mi/hr} - 30\text{ mi/hr}}{2\text{ s}} \\ &= \frac{10\text{ mi/hr}}{2\text{ s}} \cdot \frac{3600\text{ s}}{1\text{ hr}} \\ &= 18{,}000\text{ mi/hr}^2\text{ east}\end{aligned}$$

This may seem like an outrageously large number, but in terms of meters per second squared, the standard units for measuring acceleration, it comes out to 0.22 m/s^2.

Average Velocity: One Way

After reaching the tall, black ETS skyscraper, Andrea notes that the test center is exactly 10 miles from her home and that it took her precisely 16 minutes to travel between the two locations. She does a quick calculation to determine her average velocity during the trip:

$$\begin{aligned}\text{average velocity} &= \frac{10\text{ mi}}{16\text{ min}} \cdot \frac{60\text{ min}}{1\text{ hr}} \\ &= 37.5\text{ mi/hr due east}\end{aligned}$$

Average Speed and Velocity: Return Journey

Satisfied with her little exercise, Andrea turns the car around to see if she can beat her 16-minute time. Successful, she arrives home without a speeding ticket in 15 minutes. Andrea calculates her average speed for the entire journey to ETS and back home:

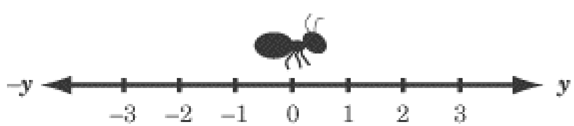
$$\begin{aligned}\text{average speed} &= \frac{20 \text{ mi}}{16 \text{ min} + 15 \text{ min}} \cdot \frac{60 \text{ min}}{1 \text{ hr}} \\ &= 38.7 \text{ mi/hr}\end{aligned}$$

Is this the same as her average velocity? Andrea reminds herself that, though her odometer reads 20 miles, her net displacement—and consequently her average velocity over the entire length of the trip—is zero. SAT II Physics is not going to get her with any trick questions like that!

Kinematics with Graphs

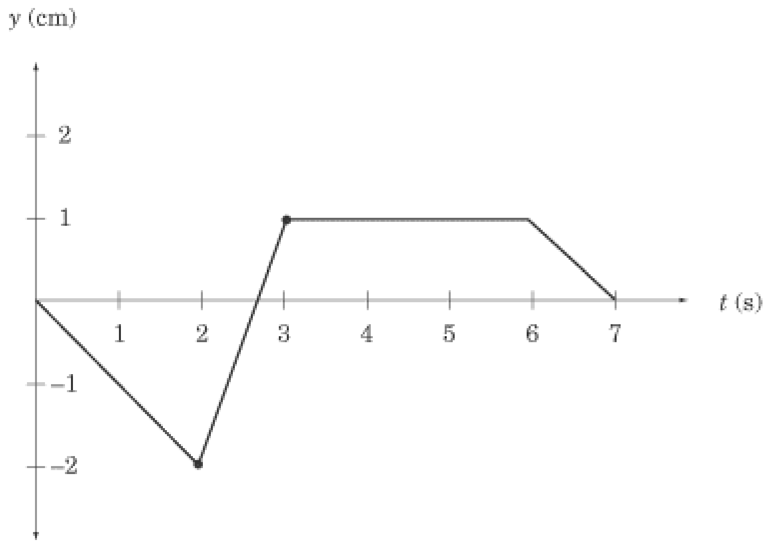
Since you are not allowed to use calculators, SAT II Physics places a heavy emphasis on qualitative problems. A common way of testing kinematics qualitatively is to present you with a graph plotting position vs. time, velocity vs. time, or acceleration vs. time and to ask you questions about the motion of the object represented by the graph. Because SAT II Physics is entirely made up of multiple-choice questions, you won't need to know how to draw graphs; you'll just have to interpret the data presented in them.

Knowing how to read such graphs quickly and accurately will not only help you solve problems of this sort, it will also help you visualize the often-abstract realm of kinematic equations. In the examples that follow, we will examine the movement of an ant running back and forth along a line.



Position vs. Time Graphs

Position vs. time graphs give you an easy and obvious way of determining an object's displacement at any given time, and a subtler way of determining that object's velocity at any given time. Let's put these concepts into practice by looking at the following graph charting the movements of our friendly ant.



Any point on this graph gives us the position of the ant at a particular moment in time. For instance, the point at $(2,-2)$ tells us that, two seconds after it started moving, the ant was two centimeters to the left of its starting position, and the point at $(3,1)$ tells us that, three seconds after it started moving, the ant is one centimeter to the right of its starting position.

Let's read what the graph can tell us about the ant's movements. For the first two seconds, the ant is moving to the left. Then, in the next second, it reverses its direction and moves quickly to $y = 1$. The ant then stays still at $y = 1$ for three seconds before it turns left again and moves back to where it started. Note how concisely the graph displays all this information.

Calculating Velocity

We know the ant's displacement, and we know how long it takes to move from place to place. Armed with this information, we should also be able to determine the ant's velocity, since velocity measures the rate of change of displacement over time. If displacement is given here by the vector **y**, then the velocity of the ant is

$$v = \frac{\Delta y}{\Delta t}$$

If you recall, the slope of a graph is a measure of rise over run; that is, the amount of change in the *y* direction divided by the amount of change in the *x* direction. In our graph, Δy is the change in the *y* direction and Δt is the change in the *x* direction, so **v** is a measure of the slope of the graph. *For any position vs. time graph, the velocity at time* t *is equal to the slope of the line at* t. In a graph made up of straight lines, like the one above, we can easily calculate the slope at each point on the graph, and hence know the instantaneous velocity at any given time.

We can tell that the ant has a velocity of zero from $t = 3$ to $t = 6$, because the slope of the line at these points is zero. We can also tell that the ant is cruising along at the fastest speed between $t = 2$ and $t = 3$, because the position vs. time graph is steepest between these points. Calculating the ant's average velocity during this time interval is a simple matter of dividing rise by run, as we've learned in math class.

$$\begin{aligned} \text{velocity} &= \frac{y_{\text{final}} - y_{\text{initial}}}{t_{\text{final}} - t_{\text{initial}}} \\ &= \frac{1(-2) \text{ cm}}{3-2 \text{ s}} \\ &= 3 \text{ cm/s to the right} \end{aligned}$$

Average Velocity

How about the average velocity between $t = 0$ and $t = 3$? It's actually easier to sort this out with a graph in front of us, because it's easy to see the displacement at $t = 0$ and $t = 3$, and so that we don't confuse displacement and distance.

$$\begin{aligned} \text{acceleration} &= \frac{\text{change in displacement}}{\text{elapsed time}} \\ &= \frac{1 - 0 \text{ cm}}{3 - 0 \text{ s}} \\ &= 0.33 \text{ cm/s to the right} \end{aligned}$$

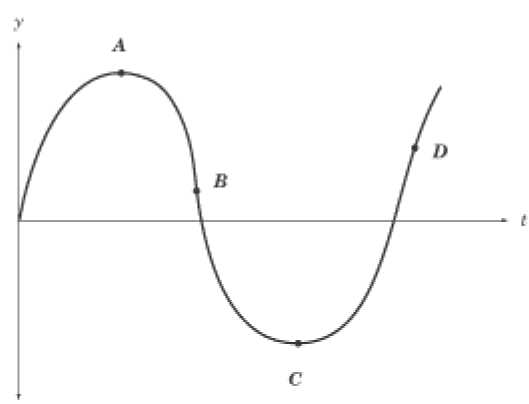
Average Speed

Although the total displacement in the first three seconds is one centimeter to the right, the total distance traveled is two centimeters to the left, and then three centimeters to the right, for a grand total of five centimeters. Thus, the average speed is not the same as the average velocity of the ant. Once we've calculated the total distance traveled by the ant, though, calculating its average speed is not difficult:

$$\frac{5 \text{ cm}}{3 \text{ s}} = 1.67 \text{ cm/s}$$

Curved Position vs. Time Graphs

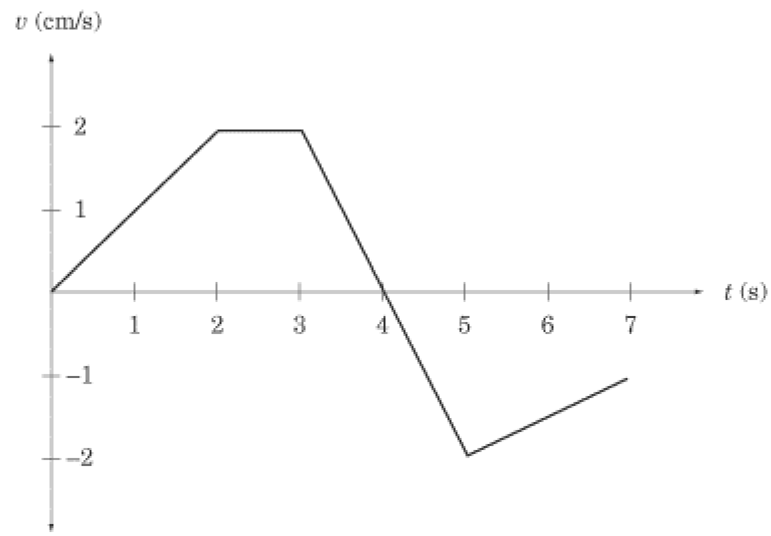
This is all well and good, but how do you calculate the velocity of a curved position vs. time graph? Well, the bad news is that you'd need calculus. The good news is that SAT II Physics doesn't expect you to use calculus, so if you are given a curved position vs. time graph, you will only be asked qualitative questions and won't be expected to make any calculations. A few points on the graph will probably be labeled, and you will have to identify which point has the greatest or least velocity. Remember, the point with the greatest slope has the greatest velocity, and the point with the least slope has the least velocity. The turning points of the graph, the tops of the "hills" and the bottoms of the "valleys" where the slope is zero, have zero velocity.



In this graph, for example, the velocity is zero at points A and C, greatest at point D, and smallest at point B. The velocity at point B is smallest because the slope at that point is negative. Because velocity is a vector quantity, the velocity at B would be a large negative number. However, the speed at B is greater even than the speed at D: speed is a scalar quantity, and so it is always positive. The slope at B is even steeper than at D, so the speed is greatest at B.

Velocity vs. Time Graphs

Velocity vs. time graphs are the most eloquent kind of graph we'll be looking at here. They tell us very directly what the velocity of an object is at any given time, and they provide subtle means for determining both the position and acceleration of the same object over time. The "object" whose velocity is graphed below is our ever-industrious ant, a little later in the day.



We can learn two things about the ant's velocity by a quick glance at the graph. First, we can tell exactly how fast it is going at any given time. For instance, we can see that, two seconds after it started to move, the ant is moving at 2 cm/s. Second, we can tell in which direction the ant is moving. From $t = 0$ to $t = 4$, the velocity is positive, meaning that the ant is moving to the right. From $t = 4$ to $t = 7$, the velocity is negative, meaning that the ant is moving to the left.

Calculating Acceleration

We can calculate acceleration on a velocity vs. time graph in the same way that we calculate velocity on a position vs. time graph. Acceleration is the rate of change of the velocity vector, $\Delta v / \Delta t$, which expresses itself as the slope of the velocity vs. time graph. For a velocity vs. time graph, *the acceleration at time* t *is equal to the slope of the line at* t.

What is the acceleration of our ant at $t = 2.5$ and $t = 4$? Looking quickly at the graph, we see that the slope of the line at $t = 2.5$ is zero and hence the acceleration is likewise zero. The slope of the graph between $t = 3$ and $t = 5$ is constant, so we can calculate the acceleration at $t = 4$ by calculating the average acceleration between $t = 3$ and $t = 5$:

$$\text{acceleration} = \frac{v_{\text{final}} - v_{\text{initial}}}{t_{\text{final}} - t_{\text{initial}}}$$
$$= \frac{-2 - (2) \text{ cm/s}}{5 - 3 \text{ s}}$$
$$= -2 \text{ cm/s}^2$$

The minus sign tells us that acceleration is in the leftward direction, since we've defined the *y*-coordinates in such a way that right is positive and left is negative. At $t = 3$, the ant is moving to the right at 2 cm/s, so a leftward acceleration means that the ant begins to slow down. Looking at the graph, we can see that the ant comes to a stop at $t = 4$, and then begins accelerating to the right.

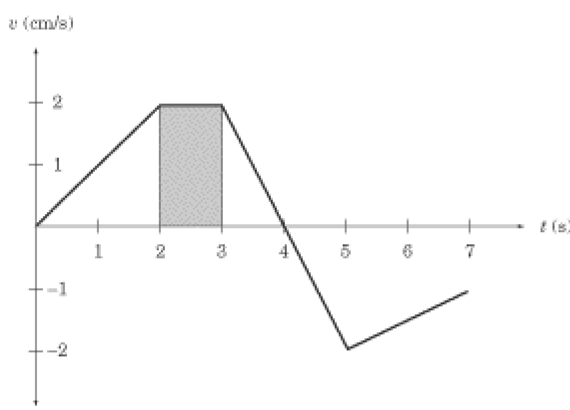
Calculating Displacement

Velocity vs. time graphs can also tell us about an object's displacement. Because velocity is a measure of displacement over time, we can infer that:

$$\text{displacement} = \text{velocity} \times \text{time}$$

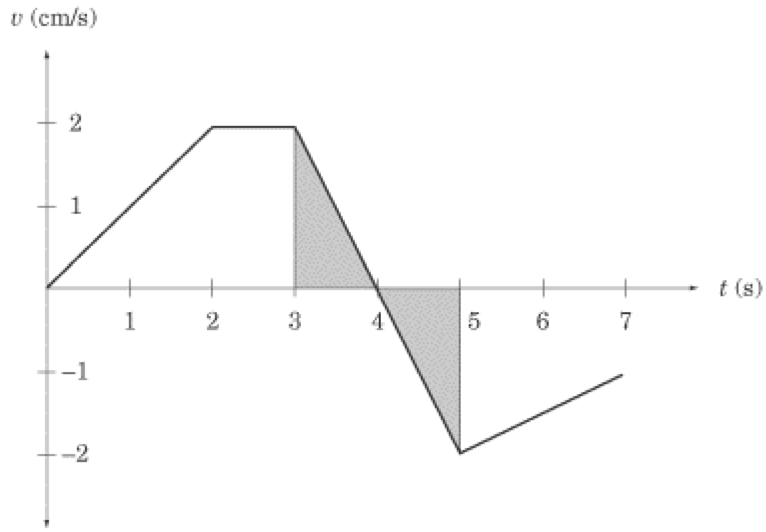
Graphically, this means that *the displacement in a given time interval is equal to the area under the graph during that same time interval.* If the graph is above the *t*-axis, then the positive displacement is the area between the graph and the *t*-axis. If the graph is below the *t*-axis, then the displacement is negative, and is the area between the graph and the *t*-axis. Let's look at two examples to make this rule clearer.

First, what is the ant's displacement between $t = 2$ and $t = 3$? Because the velocity is constant during this time interval, the area between the graph and the *t*-axis is a rectangle of width 1 and height 2.



The displacement between $t = 2$ and $t = 3$ is the area of this rectangle, which is 1 cm/s × 2 s = 2 cm to the right.

Next, consider the ant's displacement between $t = 3$ and $t = 5$. This portion of the graph gives us two triangles, one above the *t*-axis and one below the *t*-axis.



Both triangles have an area of $^1/_2(1\ \text{s})(2\ \text{cm/s}) = 1$ cm. However, the first triangle is above the t-axis, meaning that displacement is positive, and hence to the right, while the second triangle is below the t-axis, meaning that displacement is negative, and hence to the left. The total displacement between $t = 3$ and $t = 5$ is:

$$1\text{cm} - 1\text{cm} = 0$$

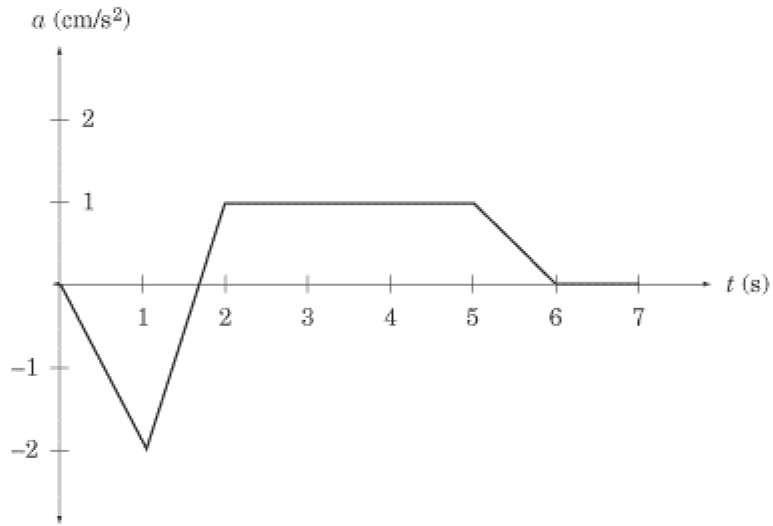
In other words, at $t = 5$, the ant is in the same place as it was at $t = 3$.

Curved Velocity vs. Time Graphs

As with position vs. time graphs, velocity vs. time graphs may also be curved. Remember that regions with a steep slope indicate rapid acceleration or deceleration, regions with a gentle slope indicate small acceleration or deceleration, and the turning points have zero acceleration.

Acceleration vs. Time Graphs

After looking at position vs. time graphs and velocity vs. time graphs, acceleration vs. time graphs should not be threatening. Let's look at the acceleration of our ant at another point in its dizzy day.

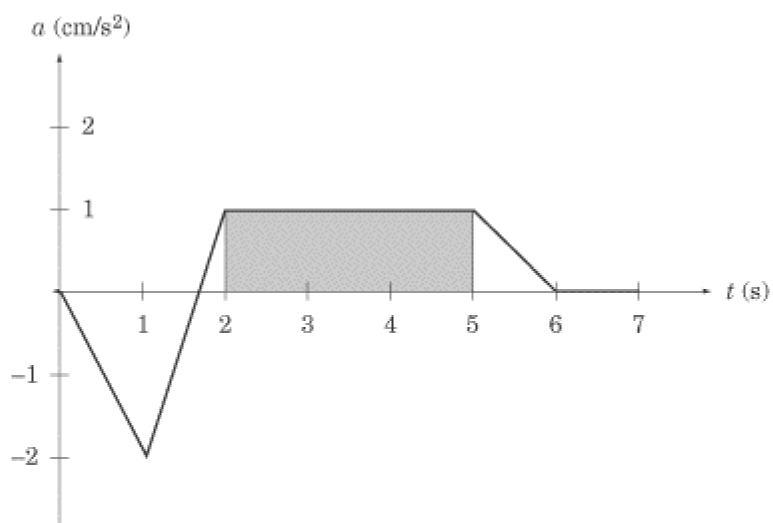


Acceleration vs. time graphs give us information about acceleration and about velocity. SAT II Physics generally sticks to problems that involve a constant acceleration. In this graph, the ant is accelerating at 1 m/s^2 from $t = 2$ to $t = 5$ and is not accelerating between $t = 6$ and $t = 7$; that is, between $t = 6$ and $t = 7$ the ant's velocity is constant.

Calculating Change in Velocity

Acceleration vs. time graphs tell us about an object's velocity in the same way that velocity vs. time graphs tell us about an object's displacement. *The change in velocity in a given time interval is equal to the area under the graph during that same time interval*. Be careful: the area between the graph and the t-axis gives the *change* in velocity, not the final velocity or average velocity over a given time period.

What is the ant's change in velocity between $t = 2$ and $t = 5$? Because the acceleration is constant during this time interval, the area between the graph and the t-axis is a rectangle of height 1 and length 3.



The area of the shaded region, and consequently the change in velocity during this time interval, is $1 \text{ cm/s}^2 \cdot 3 \text{ s} = 3 \text{ cm/s}$ to the right. This doesn't mean that the velocity at $t = 5$ is 3 cm/s; it simply means that the velocity is 3 cm/s greater than it was at $t = 2$. Since we have not been given the velocity at $t = 2$, we can't immediately say what the velocity is at $t = 5$.

Summary of Rules for Reading Graphs

You may have trouble recalling when to look for the slope and when to look for the area under the graph. Here are a couple handy rules of thumb:

1. The slope on a given graph is equivalent to the quantity we get by dividing the *y*-axis by the *x*-axis. For instance, the *y*-axis of a position vs. time graph gives us displacement, and the *x*-axis gives us time. Displacement divided by time gives us velocity, which is what the slope of a position vs. time graph represents.
2. The area under a given graph is equivalent to the quantity we get by multiplying the *x*-axis and the *y*-axis. For instance, the *y*-axis of an acceleration vs. time graph gives us acceleration, and the *x*-axis gives us time. Acceleration multiplied by time gives us the change in velocity, which is what the area between the graph and the *x*-axis represents.

We can summarize what we know about graphs in a table:

Graph	Slope	Area under the graph
position vs. time	velocity	-----
velocity vs. time	acceleration	displacement
acceleration vs. time	-----	change in velocity

One-Dimensional Motion with Uniform Acceleration

Many introductory physics problems can be simplified to the special case of uniform motion in one dimension with constant acceleration. That is, most problems will involve objects moving in a straight line whose acceleration doesn't change over time. For such problems, there are five variables that are potentially relevant: the object's position, x; the object's initial velocity, v_0; the object's final velocity, v; the object's acceleration, a; and the elapsed time, t. If you know any three of these variables, you can solve for a fourth. Here are the five **kinematic equations** that you should memorize and hold dear to your heart:

$$x = x_0 + \frac{1}{2}(v + v_0)t$$
$$v = v_0 + at$$
$$x = x_0 + v_0 t + \frac{1}{2}at^2$$
$$x = x_0 + vt - \frac{1}{2}at^2$$
$$v^2 = v_0^2 + 2a(x - x_0)$$

The variable x_0 represents the object's position at $t = 0$. Usually, $x_0 = 0$.

You'll notice there are five equations, each of which contain four of the five variables we mentioned above. In the first equation, a is missing; in the second, x is missing; in the third, v is missing; in the fourth, v_0 is missing; and in the fifth, t is missing. You'll find that in any kinematics problem, you will know three of the five variables, you'll have to solve for a fourth, and the fifth will play no role in the problem. That means you'll have to choose the equation that doesn't contain the variable that is irrelevant to the problem.

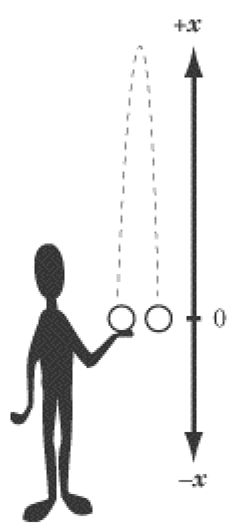
Learning to Read Verbal Clues

Problems will often give you variables like t or x, and then give you verbal clues regarding velocity and acceleration. You have to learn to translate such phrases into kinematics-equation-speak:

When They Say . . .	They Mean . . .
". . . starts from rest . . ."	$v_0 = 0$
". . . moves at a constant velocity . . ."	$a = 0$
". . . comes to rest . . . "	$v = 0$

Very often, problems in kinematics on SAT II Physics will involve a body falling under the influence of gravity. You'll find people throwing balls over their heads, at targets, and even off the Leaning Tower of Pisa. Gravitational motion is uniformly accelerated motion: the only acceleration involved is the constant pull of gravity, -9.8 m/s^2 toward the center of the Earth. When dealing with this constant, called g, it is often convenient to round it off to -10 m/s^2.

EXAMPLE



A student throws a ball up in the air with an initial velocity of 12 m/s and then catches it as it comes back down to him. What is the ball's velocity when he catches it? How high does the ball travel? How long does it take the ball to reach its highest point?

Before we start writing down equations and plugging in numbers, we need to choose a coordinate system. This is usually not difficult, but it is vitally important. Let's make the origin of the system the point where the ball is released from the student's hand and begins its upward journey, and take the up direction to be positive and the down direction to be negative.

We could have chosen other coordinate systems—for instance, we could have made the origin the ground on which the student is standing—but our choice of coordinate system is convenient because in it, $x_0 = 0$, so we won't have to worry about plugging a value for x_0 into our equation. It's usually possible, and a good idea, to choose a coordinate system that eliminates x_0. Choosing the up direction as positive is simply more intuitive, and thus less likely to lead us astray. It's generally wise also to choose your coordinate system so that more variables will be positive numbers than negative ones, simply because positive numbers are easier to deal with.

WHAT IS THE BALL'S VELOCITY WHEN HE CATCHES IT?

We can determine the answer to this question without any math at all. We know the initial velocity, $v_0 = 12$ m/s, and the acceleration due to gravity, $a = -10$ m/s², and we know that the displacement is $x = 0$ since the ball's final position is back in the student's hand where it started. We need to know the ball's final velocity, v, so we should look at the kinematic equation that leaves out time, t:

$$v^2 = v_0^2 + 2a(x - x_0)$$

Because both x and x_0 are zero, the equation comes out to $v^2 = v_0^2$. But don't be hasty and give the answer as 12 m/s: remember that we devised our coordinate system in such a way that the down direction is negative, so the ball's final velocity is –12 m/s.

HOW HIGH DOES THE BALL TRAVEL?

We know that at the top of the ball's trajectory its velocity is zero. That means that we know that $v_0 = 12$ m/s, $v = 0$, and $a = -10$ m/s², and we need to solve for x:

$$\begin{aligned} v^2 &= v_0^2 + 2a(x - x_0) \\ x &= \frac{c^2 - v_0^2}{2a} \\ &= \frac{-(-12 \text{ m/s}^2)}{(2)(-10 \text{ m/s}^2)} \\ &= 7.2 \text{ m} \end{aligned}$$

HOW LONG DOES IT TAKE THE BALL TO REACH ITS HIGHEST POINT?

Having solved for x at the highest point in the trajectory, we now know all four of the other variables related to this point, and can choose any one of the five equations to solve for t. Let's choose the one that leaves out x:

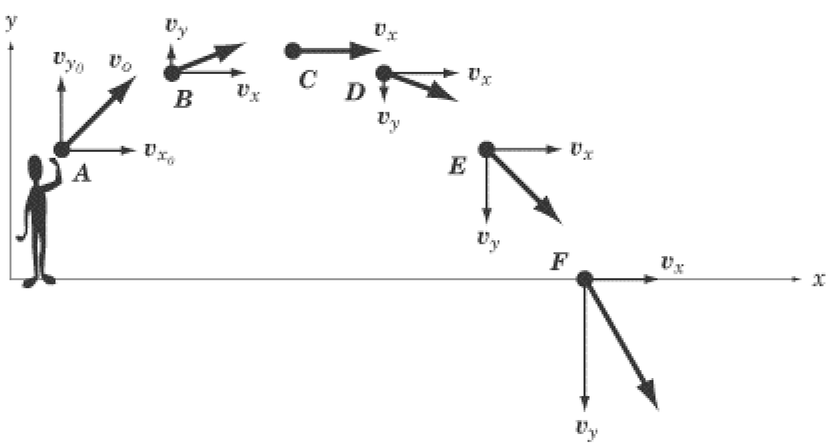
$$v = v_0 + at$$
$$t = \frac{v - v_0}{a}$$
$$= \frac{-12\text{m/s}}{-10\text{m/s}^2}$$
$$= 1.2\text{s}$$

Note that there are certain convenient points in the ball's trajectory where we can extract a third variable that isn't mentioned explicitly in the question: we know that $x = 0$ when the ball is at the level of the student's hand, and we know that $v = 0$ at the top of the ball's trajectory.

Two-Dimensional Motion with Uniform Acceleration

If you've got the hang of 1-D motion, you should have no trouble at all with 2-D motion. The motion of any object moving in two dimensions can be broken into *x*- and *y*-components. Then it's just a matter of solving two separate 1-D kinematic equations.

The most common problems of this kind on SAT II Physics involve projectile motion: the motion of an object that is shot, thrown, or in some other way launched into the air. Note that the motion or trajectory of a projectile is a parabola.



If we break this motion into *x*- and *y*-components, the motion becomes easy to understand. In the *y* direction, the ball is thrown upward with an initial velocity of v_{y_0} and experiences a constant downward acceleration of $g = -9.8$ m/s². This is exactly the kind of motion we examined in the previous section: if we ignore the *x*-component, the motion of a projectile is identical to the motion of an object thrown directly up in the air.

In the *x* direction, the ball is thrown forward with an initial velocity of v_{x_0} and there is no acceleration acting in the *x* direction to change this velocity. We have a very simple situation where $a_x = 0$ and v_x is constant.

SAT II Physics will probably not expect you to do much calculating in questions dealing with projectile motion. Most likely, it will ask about the relative velocity of the projectile at different points in its trajectory. We can calculate the *x*- and *y*-components separately and then combine them to find the velocity of the projectile at any given point:

$$v = \sqrt{v_x^2 + v_y^2}$$

Because v_x is constant, the speed will be greater or lesser depending on the magnitude of v_y. To determine where the speed is least or greatest, we follow the same method as we would with the one-dimensional example we had in the previous section. That means that the speed of the projectile in the figure above is at its greatest at position *F*, and at its least at position *C*. We also know that the speed is equal at position *B* and position *D*, and at position *A* and position *E*. The key with two-dimensional motion is to remember that you are not dealing with one complex equation of motion, but rather with two simple equations.

Key Formulas

Average Speed	average speed = $\frac{\text{distance traveled}}{\text{time elapsed}} = \frac{\Delta x}{\Delta t}$
Average Velocity	average velocity = $\frac{\text{change in displacement}}{\text{time elapsed}} = \frac{\Delta s}{\Delta t}$
Average Acceleration	average acceleration = $\frac{\text{change in velocity}}{\text{time elapsed}} = \frac{\Delta v}{\Delta t}$
One-Dimensional Motion with Uniform Acceleration (a.k.a. "The Five Kinematic Equations")	$x = x_0 + \frac{1}{2}(v + v_0)t$ $v = v_0 + at$ $x = x_0 + v_0 t + \frac{1}{2}at^2$ $x = x_0 + vt - \frac{1}{2}at^2$ $v^2 = v_0^2 + 2a(x - x_0)$
Velocity of Two-Dimensional Projectiles	$v = \sqrt{v_x^2 + v_y^2}$

Practice Questions

1. An athlete runs four laps of a 400 m track. What is the athlete's total displacement?

   (A) –1600 m

   (B) –400 m

   (C) 0 m

   (D) 400 m

   (E) 1600 m

2. Which of the following statements contains a reference to displacement?

   I. "The town is a five mile drive along the winding country road."
   II. "The town sits at an altitude of 940 m."
   III. "The town is ten miles north, as the crow flies."

   (A) I only

   (B) III only

   (C) I and III only

   (D) II and III only

   (E) I, II, and III

Questions 3 and 4 refer to a car that travels from point *A* to point *B* in four hours, and then from point *B* back to point *A* in six hours. The road between point *A* and point *B* is perfectly straight, and the distance between the two points is 240 km.

3. What is the car's average velocity?

   (A) 0 km/h

   (B) 48 km/h

   (C) 50 km/h

   (D) 60 km/h

   (E) 100 km/h

4. What is the car's average speed?

   (A) 0 km/h

   (B) 48 km/h

   (C) 50 km/h

   (D) 60 km/h

   (E) 100 km/h

5. A ball is dropped from the top of a building. Taking air resistance into account, which best describes the speed of the ball while it is moving downward?

   (A) It will increase until it reaches the speed of light

   (B) It will increase at a steady rate

   (C) It will remain constant

   (D) It will decrease

   (E) Its rate of acceleration will decrease until the ball moves at a constant speed

6. A car accelerates steadily so that it goes from a velocity of 20 m/s to a velocity of 40 m/s in 4 seconds. What is its acceleration?

   (A) 0.2 m/s^2

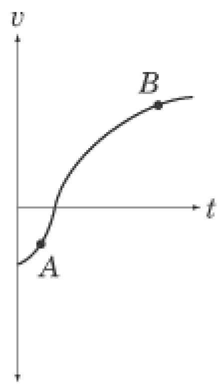
   (B) 4 m/s^2

   (C) 5 m/s^2

   (D) 10 m/s^2

   (E) 80 m/s^2

Questions 7 and 8 relate to the graph of velocity vs. time of a moving particle plotted at right.



7. What is the acceleration and displacement of the particle at point *A*?

   (A) Acceleration decreasing, displacement decreasing

   (B) Acceleration constant, displacement decreasing

   (C) Acceleration increasing, displacement decreasing

   (D) Acceleration decreasing, displacement increasing

   (E) Acceleration increasing, displacement increasing

8. How do the acceleration and displacement of the particle at point *B* compare to the acceleration and displacement of the particle at point *A*?

   (A) Acceleration is less, displacement is less

   (B) Acceleration is less, displacement is the same

   (C) Acceleration is less, displacement is greater

   (D) Acceleration is greater, displacement is less

   (E) Acceleration is greater, displacement is greater

9. A sprinter starts from rest and accelerates at a steady rate for the first 50 m of a 100 m race, and then continues at a constant velocity for the second 50 m of the race. If the sprinter runs the 100 m in a time of 10 s, what is his instantaneous velocity when he crosses the finish line?

   (A) 5 m/s

   (B) 10 m/s

   (C) 12 m/s

   (D) 15 m/s

   (E) 20 m/s

10. A woman runs 40 m to the north in 6.0 s, and then 30 m to the east in 4.0Â s. What is the magnitude of her average velocity?

(A) 5.0 m/s

(B) 6.0 m/s

(C) 6.7 m/s

(D) 7.0 m/s

(E) 7.5 m/s

Dynamics

WHEREAS KINEMATICS IS THE STUDY OF objects in motion, **dynamics** is the study of the *causes* of motion. In other words, kinematics covers the “what” of motion, while dynamics covers the “how” and “why.” **Forces** are the lifeblood of dynamics: objects move and change their motion under the influence of different forces. Our main emphasis will be on Newton’s three laws, which succinctly summarize everything you need to know about dynamics.

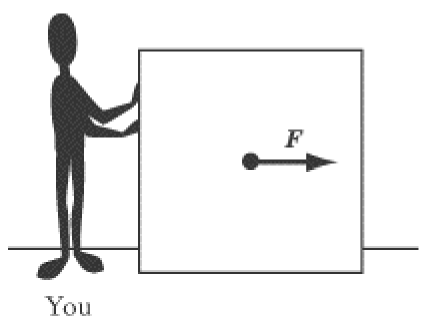
Dynamics questions on SAT II Physics often call upon your knowledge of kinematics and vectors, but these questions will probably be simpler than the problems you’ve encountered in your physics class. Because you won’t be asked to do any math that would require a calculator, you should focus on mastering the concepts that lie behind the math.

What Are Forces?

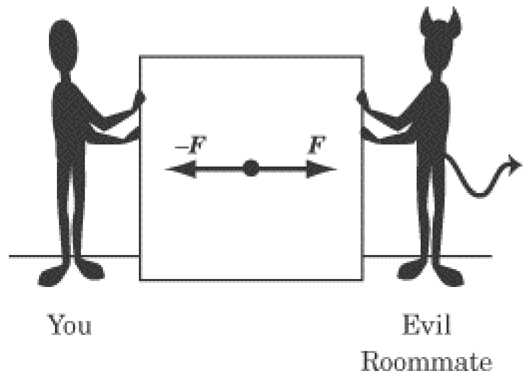
Whenever we lift something, push something, or otherwise manipulate an object, we are exerting a force. A force is defined very practically as a push or a pull—essentially it’s what makes things move. A force is a vector quantity, as it has both a magnitude and a direction.

In this chapter, we will use the example of pushing a box along the floor to illustrate many concepts about forces, with the assumption that it’s a pretty intuitive model that you will have little trouble imagining.

Physicists use simple pictures called **free-body diagrams** to illustrate the forces acting on an object. In these diagrams, the forces acting *on* a body are drawn as vectors originating from the center of the object. Following is a free-body diagram of you pushing a box into your new college dorm with force $\boldsymbol{F}$.



Because force is a vector quantity, it follows the rules of vector addition. If your evil roommate comes and pushes the box in the opposite direction with exactly the same magnitude of force (force $-\boldsymbol{F}$), the net force on the box is zero



Newton's Laws

Isaac Newton first published his three laws of motion in 1687 in his monumental *Mathematical Principles of Natural Philosophy*. In these three simple laws, Newton sums up everything there is to know about dynamics. This achievement is just one of the many reasons why he is considered one of the greatest physicists in history.

While a multiple-choice exam can't ask you to write down each law in turn, there is a good chance you will encounter a problem where you are asked to choose which of Newton's laws best explains a given physical process. You will also be expected to make simple calculations based on your knowledge of these laws. But by far the most important reason for mastering Newton's laws is that, without them, thinking about dynamics is impossible. For that reason, we will dwell at some length on describing how these laws work qualitatively.

Newton's First Law

Newton's First Law describes how forces relate to motion:

An object at rest remains at rest, unless acted upon by a net force. An object in motion remains in motion, unless acted upon by a net force.

A soccer ball standing still on the grass does not move until someone kicks it. An ice hockey puck will continue to move with the same velocity until it hits the boards, or someone else hits it. Any change in the velocity of an object is evidence of a net force acting on that object. A world without forces would be much like the images we see of the insides of spaceships, where astronauts, pens, and food float eerily about.

Remember, since velocity is a vector quantity, a change in velocity can be a change either in the magnitude or the direction of the velocity vector. A moving object upon which no net force is acting doesn't just maintain a constant speed—it also moves in a straight line.

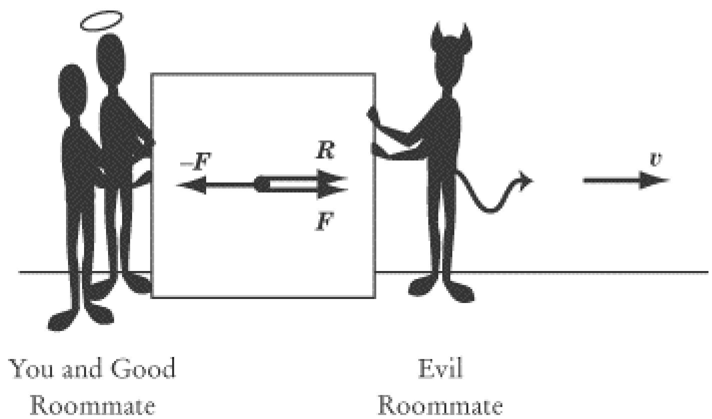
But what does Newton mean by a *net* force? The net force is the sum of the forces acting on a body. Newton is careful to use the phrase "net force," because an object at rest will stay at rest if acted upon by forces with a sum of zero. Likewise, an object in motion will retain a constant velocity if acted upon by forces with a sum of zero.

Consider our previous example of you and your evil roommate pushing with equal but opposite forces on a box. Clearly, force is being applied to the box, but the two forces on the box cancel each other out exactly: $\mathbf{F} + -\mathbf{F} = 0$. Thus the net force on the box is zero, and the box does not move.

Yet if your other, good roommate comes along and pushes alongside you with a force $\boldsymbol{R}$, then the tie will be broken and the box will move. The net force is equal to:

$$\boldsymbol{F} + -\boldsymbol{F} + \boldsymbol{R} = \boldsymbol{R}$$

Note that the acceleration, $\boldsymbol{a}$, and the velocity of the box, $\boldsymbol{v}$, is in the same direction as the net force.



Inertia

The First Law is sometimes called the law of **inertia**. We define inertia as the tendency of an object to remain at a constant velocity, or its resistance to being accelerated. Inertia is a fundamental property of all matter and is important to the definition of **mass**.

Newton's Second Law

To understand **Newton's Second Law**, you must understand the concept of mass. Mass is an intrinsic scalar quantity: it has no direction and is a property of an object, not of the object's location. Mass is a measurement of a body's inertia, or its resistance to being accelerated. The words *mass* and *matter* are related: a handy way of thinking about mass is as a measure of how much matter there is in an object, how much "stuff" it's made out of. Although in everyday language we use the words *mass* and *weight* interchangeably, they refer to two different, but related, quantities in physics. We will expand upon the relation between mass and weight later in this chapter, after we have finished our discussion of Newton's laws.

We already have some intuition from everyday experience as to how mass, force, and acceleration relate. For example, we know that the more force we exert on a bowling ball, the faster it will roll. We also know that if the same force were exerted on a basketball, the basketball would move faster than the bowling ball because the basketball has less mass. This intuition is quantified in Newton's Second Law:

$$\boldsymbol{F} = m\boldsymbol{a}$$

Stated verbally, Newton's Second Law says that the net force, $\boldsymbol{F}$, acting on an object causes the object to accelerate, $\boldsymbol{a}$. Since $\boldsymbol{F} = m\boldsymbol{a}$ can be rewritten as $\boldsymbol{a} = \boldsymbol{F}/m$, you can see that the magnitude of the acceleration is directly proportional to the net force and inversely proportional to the mass, m. Both force and acceleration are vector quantities, and the acceleration of an object will always be in the *same* direction as the net force.

The unit of force is defined, quite appropriately, as a **newton** (N). Because acceleration is given in units of m/s^2 and mass is given in units of kg, Newton's Second Law implies that $1 \text{ N} = 1 \text{ kg} \cdot \text{m/s}^2$. In other words, one newton is the force required to accelerate a one-kilogram body, by one meter per second, each second.

Newton's Second Law in Two Dimensions

With a problem that deals with forces acting in two dimensions, the best thing to do is to break each force vector into its *x*- and *y*-components. This will give you two equations instead of one:

$$\boldsymbol{F}_x = m\boldsymbol{a}_x$$
$$\boldsymbol{F}_y = m\boldsymbol{a}_y$$

The component form of Newton's Second Law tells us that the component of the net force in the x direction is directly proportional to the resulting component of the acceleration in the x direction, and likewise for the *y*-component.

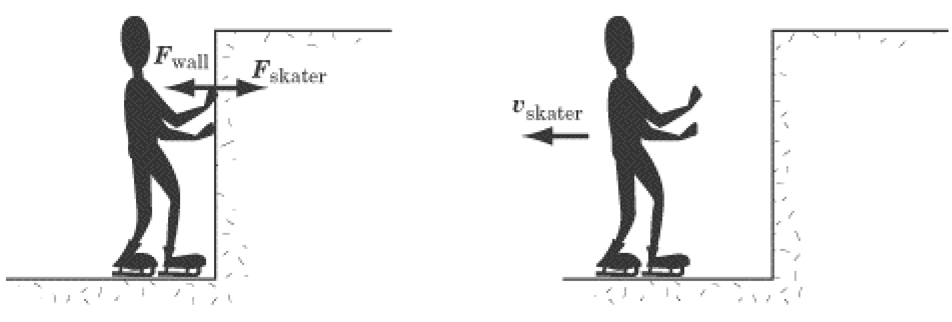
Newton's Third Law

Newton's Third Law has become a cliché. The Third Law tells us that:

To every action, there is an equal and opposite reaction.

What this tells us in physics is that every push or pull produces not one, but two forces. In any exertion of force, there will always be two objects: the object exerting the force and the object on which the force is exerted. Newton's Third Law tells us that when object A exerts a force $\boldsymbol{F}$ on object B, object B will exert a force $-\boldsymbol{F}$ on object A. When you push a box forward, you also feel the box pushing back on your hand. If Newton's Third Law did not exist, your hand would feel nothing as it pushed on the box, because there would be no reaction force acting on it.

Anyone who has ever played around on skates knows that when you push forward on the wall of a skating rink, you recoil backward.



skater pushes on wall

skater recoils off wall

Newton's Third Law tells us that the force that the skater exerts on the wall, $\boldsymbol{F}_{\text{skater}}$, is exactly equal in magnitude and opposite in direction to the force that the wall exerts on the skater, $\boldsymbol{F}_{\text{wall}}$. The harder the skater pushes on the wall, the harder the wall will push back, sending the skater sliding backward.

Newton's Third Law at Work

Here are three other examples of Newton's Third Law at work, variations of which often pop up on SAT II Physics:

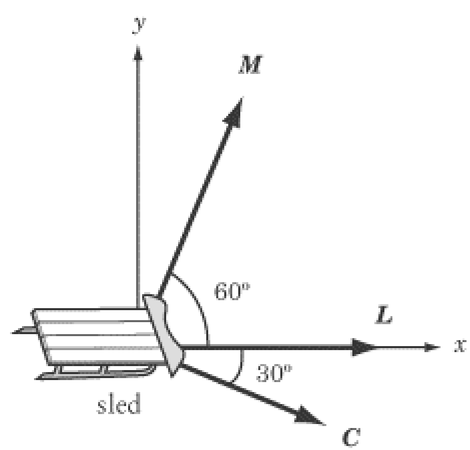
- You push down with your hand on a desk, and the desk pushes upward with a force equal in magnitude to your push.
- A brick is in free fall. The brick pulls the Earth upward with the same force that the Earth pulls the brick downward.
- When you walk, your feet push the Earth backward. In response, the Earth pushes your feet forward, which is the force that moves you on your way.

The second example may seem odd: the Earth doesn't move upward when you drop a brick. But recall Newton's Second Law: the acceleration of an object is inversely proportional to its mass ($\mathbf{a} = \mathbf{F}/m$). The Earth is about 10^{24} times as massive as a brick, so the brick's downward acceleration of -9.8 m/s² is about 10^{24} times as great as the Earth's upward acceleration. The brick exerts a force on the Earth, but the effect of that force is insignificant.

Problem Solving with Newton's Laws

Dynamics problem solving in physics class usually involves difficult calculations that take into account a number of vectors on a free-body diagram. SAT II Physics won't expect you to make any difficult calculations, and the test will usually include the free-body diagrams that you need. Your task will usually be to interpret free-body diagrams rather than to draw them.

EXAMPLE 1



> The Three Stooges are dragging a 10 kg sled across a frozen lake. Moe pulls with force $\mathbf{M}$, Larry pulls with force $\mathbf{L}$, and Curly pulls with force $\mathbf{C}$. If the sled is moving in the $\hat{\mathbf{x}}$ direction, and both Moe and Larry are exerting a force of 10 N, what is the magnitude of the force Curly is exerting? Assuming that friction is negligible, what is the acceleration of the sled? (Note: sin 30 = cos 60 = 0.500 and sin 60 = cos 30 = 0.866.)

The figure above gives us a free-body diagram that shows us the direction in which all forces are acting, but we should be careful to note that vectors in the diagram are not drawn to scale: we cannot estimate the magnitude of $\mathbf{C}$ simply by comparing it to $\mathbf{M}$ and $\mathbf{L}$.

What is the magnitude of the force Curly is exerting?

Since we know that the motion of the sled is in the $\hat{x}$ direction, the net force, ***M*** + ***L*** +***C***, must also be in the $\hat{x}$ direction. And since the sled is not moving in the $\hat{y}$ direction, the *y*-component of the net force must be zero. Because the *y*-component of Larry's force is zero, this implies:

$$\boldsymbol{M}_y + \boldsymbol{C}_y = 0$$

where M_y is the *y*-component of ***M*** and C_y is the *y*-component of ***C***. We also know:

$$\begin{aligned} \boldsymbol{M}_y &= M \sin\theta = (10\text{ N})\sin 60 = 8.660\text{ N} \\ \boldsymbol{C}_y &= C\sin\theta = C\sin(-30) = -0.500C \end{aligned}$$

If we substitute these two equations for M_y and C_y into the equation $M_y + C_y = 0$, we have:

$$\begin{aligned} 8.660\text{ N} - 0.500C &= 0 \\ C &= 17.32\text{ N} \end{aligned}$$

What is the acceleration of the sled?

According to Newton's Second Law, the acceleration of the sled is ***a*** = ***F***/m. We know the sled has a mass of 10 kg, so we just need to calculate the magnitude of the net force in the $\hat{x}$-direction.

$$\begin{aligned} \boldsymbol{M}_x + \boldsymbol{L}_x + \boldsymbol{C}_x &= (10\text{ N})\cos 60 + 10\text{ N} + (17.32\text{ N})\cos(-30) \\ &= 0.500(10\text{ N}) + 10\text{ N} + 0.866(17.32\text{ N}) \\ &= 30.000\text{ N} \end{aligned}$$

Now that we have calculated the magnitude of the net force acting on the sled, a simple calculation can give us the sled's acceleration:

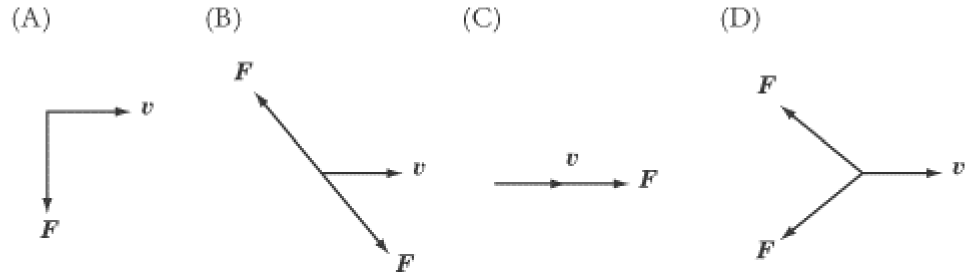
$$\begin{aligned} \boldsymbol{a} &= \frac{\boldsymbol{F}}{m} = \frac{30.000\text{ N}}{10\text{ kg}} \\ &= 3.000\text{ m/s}^2 \end{aligned}$$

We have been told that the sled is moving in the $\hat{x}$ direction, so the acceleration is also in the $\hat{x}$ direction.

This example problem illustrates the importance of vector components. For the SAT II, you will need to break vectors into components on any problem that deals with vectors that are not all parallel or perpendicular. As with this example, however, the SAT II will always provide you with the necessary trigonometric values.

EXAMPLE 2

Each of the following free-body diagrams shows the instantaneous forces, ***F***, acting on a particle and the particle's instantaneous velocity, ***v***. All forces represented in the diagrams are of the same magnitude.



1. In which diagram is neither the speed nor the direction of the particle being changed?

2. In which diagram is the speed but not the direction of the particle being changed?

3. In which diagram is the direction but not the speed of the particle being changed?

4. In which diagram are both the speed and direction of the particle being changed?

The answer to question 1 is **B**. The two forces in that diagram cancel each other out, so the net force on the particle is zero. The velocity of a particle only changes under the influence of a net force. The answer to question 2 is **C**. The net force is in the same direction as the particle's motion, so the particle continues to accelerate in the same direction. The answer to question 3 is **A**. Because the force is acting perpendicular to the particle's velocity, it does not affect the particle's speed, but rather acts to pull the particle in a circular orbit. Note, however, that the speed of the particle only remains constant if the force acting on the particle remains perpendicular to it. As the direction of the particle changes, the direction of the force must also change to remain perpendicular to the velocity. This rule is the essence of circular motion, which we will examine in more detail later in this book. The answer to question 4 is **D**. The net force on the particle is in the opposite direction of the particle's motion, so the particle slows down, stops, and then starts accelerating in the opposite direction.

Types of Forces

There are a number of forces that act in a wide variety of cases and have been given specific names. Some of these, like friction and the normal force, are so common that we're hardly aware of them as distinctive forces. It's important that you understand how and when these forces function, because questions on SAT II Physics often make no mention of them explicitly, but expect you to factor them into your calculations. Some of these forces will also play an important role in the chapter on special problems in mechanics.

Weight

Although the words weight and mass are often interchangeable in everyday language, these words refer to two different quantities in physics. The mass of an object is a property of the object itself, which reflects its resistance to being accelerated. The weight of an object is a measure of the gravitational force being exerted upon it, and so it varies depending on the gravitational force acting on the object. Mass is a scalar quantity measured in kilograms, while weight is a vector quantity measuring force, and is represented in newtons. Although an object's mass never changes, its weight depends on the force of gravity in the object's environment.

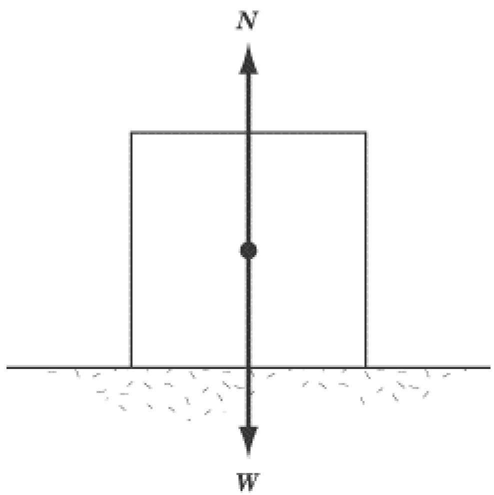
For example, a 10 kg mass has a different weight on the moon than it does on Earth. According to Newton's Second Law, the weight of a 10 kg mass on Earth is

$$\begin{aligned} \boldsymbol{F} &= m\boldsymbol{g}_{\text{earth}} \\ &= (10 \text{ kg})(9.8 \text{ m/s}^2) \\ &= 98 \text{ N} \end{aligned}$$

This force is directed toward the center of the Earth. On the moon, the acceleration due to gravity is roughly one-sixth that on Earth. Therefore, the weight of a 10 kg mass on the moon is only about 16.3 N toward the center of the moon.

The Normal Force

The **normal force** always acts perpendicular (or "normal") to the surface of contact between two objects. The normal force is a direct consequence of Newton's Third Law. Consider the example of a 10 kg box resting on the floor. The force of gravity causes the box to push down upon the ground with a force, $\boldsymbol{W}$, equal to the box's weight. Newton's Third Law dictates that the floor must apply an equal and opposite force, $\boldsymbol{N} = -\boldsymbol{W}$, to the box. As a result, the net force on the box is zero, and, as we would expect, the box remains at rest. If there were no normal force pushing the box upward, there would be a net force acting downward on the box, and the box would accelerate downward

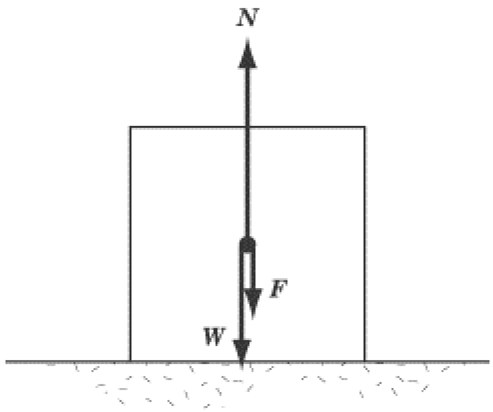


Be careful not to confuse the normal force vector $\boldsymbol{N}$ with the abbreviation for newtons, N. It can be a bit confusing that both are denoted by the same letter of the alphabet, but they are two totally different entities.

EXAMPLE

> A person pushes downward on a box of weight $\boldsymbol{W}$ with a force $\boldsymbol{F}$. What is the normal force, $\boldsymbol{N}$, acting on the box?

The total force pushing the box toward the ground is $\boldsymbol{W} + \boldsymbol{F}$. From Newton's Third Law, the normal force exerted on the box by the floor has the same magnitude as $\boldsymbol{W} + \boldsymbol{F}$ but is directed upward. Therefore, the net force on the box is zero and the box remains at rest.



Friction

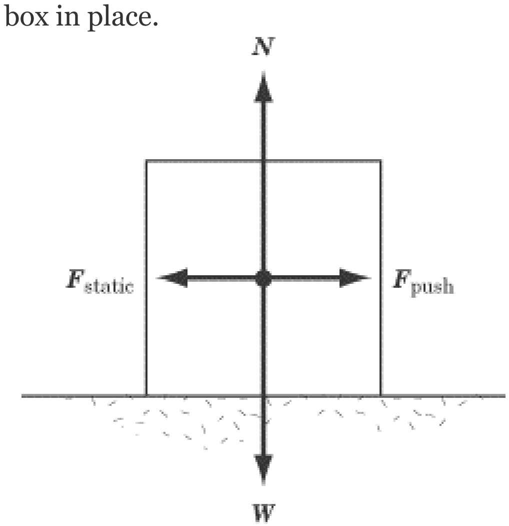
Newton's First Law tells us that objects in motion stay in motion unless a force is acting upon them, but experience tells us that when we slide coins across a table, or push boxes along the floor, they slow down and come to a stop. This is not evidence that Newton was wrong; rather, it shows that there is a force acting upon the coin or the box to slow its motion. This is the force of **friction**, which is at work in every medium but a vacuum, and is the bugbear of students pushing boxes across the sticky floors of dorm rooms everywhere.

Roughly speaking, frictional forces are caused by the roughness of the materials in contact, deformations in the materials, and molecular attraction between materials. You needn't worry too much over the causes of friction, though: SAT II Physics isn't going to test you on them. The most important thing to remember about frictional forces is that they are always parallel to the plane of contact between two surfaces, and opposite to the direction that the object is being pushed or pulled.

There are two main types of friction: **static friction** and **kinetic friction**. Kinetic friction is the force between two surfaces moving relative to one another, whereas static friction is the force between two surfaces that are not moving relative to one another.

Static Friction

Imagine, once more, that you are pushing a box along a floor. When the box is at rest, it takes some effort to get it to start moving at all. That's because the force of static friction is resisting your push and holding the box in place.



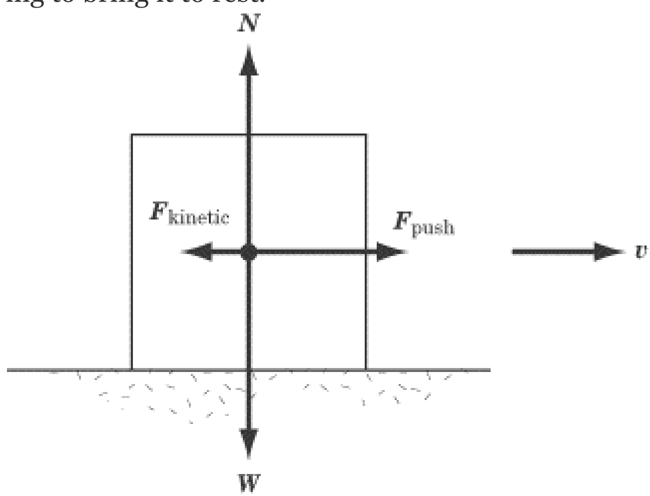
In the diagram above, the weight and the normal force are represented as W and N respectively, and the force applied to the box is denoted by F_{push}. The force of static friction is represented by F_{static}, where $F_{static} = F_{push}$. The net force on the box is zero, and so the box does not move.

This is what happens when you are pushing on the box, but not hard enough to make it budge.

Static friction is only at work when the net force on an object is zero, and hence when $F_{static} = -F_{push}$. If there is a net force on the object, then that object will be in motion, and kinetic rather than static friction will oppose its motion.

Kinetic Friction

The force of static friction will only oppose a push up to a point. Once you exert a strong enough force, the box will begin to move. However, you still have to keep pushing with a strong, steady force to keep it moving along, and the box will quickly slide to a stop if you quit pushing. That's because the force of kinetic friction is pushing in the opposite direction of the motion of the box, trying to bring it to rest.



Though the force of kinetic friction will always act in the opposite direction of the force of the push, it need not be equal in magnitude to the force of the push. In the diagram above, the magnitude of $F_{kinetic}$ is less than the magnitude of F_{push}. That means that the box has a net force in the direction of the push, and the box accelerates forward. The box is moving at velocity v in the diagram, and will speed up if the same force is steadily applied to it. If F_{push} were equal to $-F_{kinetic}$, the net force acting on the box would be zero, and the box would move at a steady velocity of v, since Newton's First Law tells us that an object in motion will remain in motion if there is no net force acting on it. If the magnitude of F_{push} were less than the magnitude of $F_{kinetic}$, the net force would be acting against the motion, and the box would slow down until it came to a rest.

The Coefficients of Friction

The amount of force needed to overcome the force of static friction on an object, and the magnitude of the force of kinetic friction on an object, are both proportional to the normal force acting on the object in question. We can express this proportionality mathematically as follows:

$$F_{\text{kinetic}} = \mu_k N = \mu_k mg$$
$$F_{\text{static, max}} = \mu_s N = \mu_s mg$$

where μ_k is the **coefficient of kinetic friction**, μ_s is the **coefficient of static friction**, and N is the magnitude of the normal force. The coefficients of kinetic and static friction are constants of proportionality that vary from object to object.

Note that the equation for static friction is for the *maximum* value of the static friction. This is because the force of static friction is never greater than the force pushing on an object. If a box has a mass of 10 kg and $\mu_s = 0.5$, then:

$$\begin{aligned} F_{\text{static, max}} &= 0.5(10\text{ kg})(9.8\text{ m/s}^2) \\ &= 49\text{ N} \end{aligned}$$

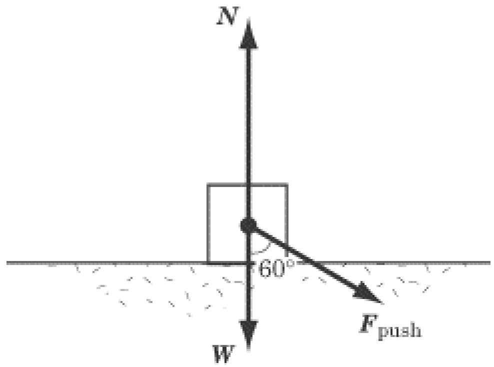
If you push this box with a force less than 49 newtons, the box will not move, and consequently the net force on the box must be zero. If an applied force F_{push} is less than $F_{\text{static, max}}$, then $F_{\text{static}} = -F_{\text{push}}$.

Three Reminders

Whenever you need to calculate a frictional force on SAT II Physics, you will be told the value of μ, which will fall between 0 and 1. Three things are worth noting about frictional forces:

1. **The smaller μ is, the more slippery the surface.** For instance, ice will have much lower coefficients of friction than Velcro. In cases where $\mu = 0$, the force of friction is zero, which is the case on ideal frictionless surfaces.
2. **The coefficient of kinetic friction is smaller than the coefficient of static friction.** That means it takes more force to start a stationary object moving than to keep it in motion. The reverse would be illogical: imagine if you could push on an object with a force greater than the maximum force of static friction but less than the force of kinetic friction. That would mean you could push it hard enough to get it to start moving, but as soon as it starts moving, the force of kinetic friction would push it backward.
3. **Frictional forces are directly proportional to the normal force.** That's why it's harder to slide a heavy object along the floor than a light one. A light coin can slide several meters across a table because the kinetic friction, proportional to the normal force, is quite small.

EXAMPLE



A student pushes a box that weighs 15 N with a force of 10 N at a 60Â° angle to the perpendicular. The maximum coefficient of static friction between the box and the floor is 0.4. Does the box move? Note that sin 60Â° = 0.866 and cos 60Â° = 0.500.

In order to solve this problem, we have to determine whether the horizontal component of $\boldsymbol{F}_{\text{push}}$ is of greater magnitude than the maximum force of static friction.

We can break the $\boldsymbol{F}_{\text{push}}$ vector into horizontal and vertical components. The vertical component will push the box harder into the floor, increasing the normal force, while the horizontal component will push against the force of static friction. First, let's calculate the vertical component of the force so that we can determine the normal force, ***N***, of the box:

$$\boldsymbol{F}_{\text{push, y}} = \boldsymbol{F}_{\text{push}} \cos 60^\circ = 0.500(10\text{ N}) = 5.0\text{ N}$$

If we add this force to the weight of the box, we find that the normal force is 15 + 5.0 = 20 N. Thus, the maximum force of static friction is:

$$\boldsymbol{F}_{\text{static, max}} = \mu_s N = 0.4(20\text{ N}) = 8.0\text{ N}$$

The force pushing the box forward is the horizontal component of $\boldsymbol{F}_{\text{push}}$, which is:

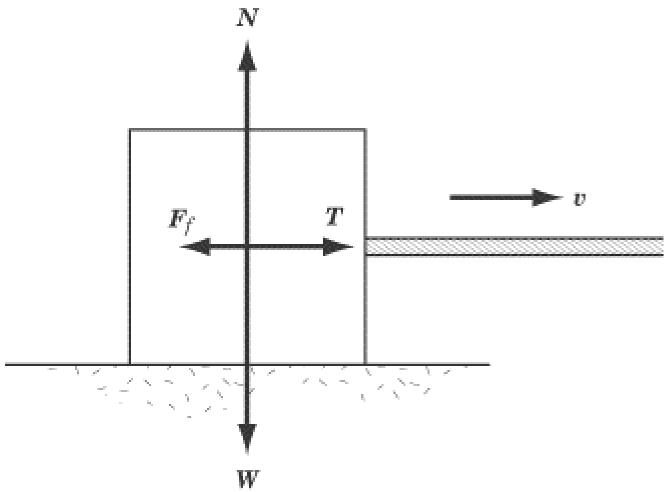
$$\boldsymbol{F}_{\text{push, x}} = F_{\text{push}} \sin 60^\circ = 0.866(10\text{ N}) = 8.66\text{ N}$$

As we can see, this force is just slightly greater than the maximum force of static friction opposing the push, so the box will slide forward.

Tension

Consider a box being pulled by a rope. The person pulling one end of the rope is not in contact with the box, yet we know from experience that the box will move in the direction that the rope is pulled. This occurs because the force the person exerts on the rope is transmitted to the box.

The force exerted on the box from the rope is called the **tension** force, and comes into play whenever a force is transmitted across a rope or a cable. The free-body diagram below shows us a box being pulled by a rope, where ***W*** is the weight of the box, *N* is the normal force, ***T*** is the tension force, and $\boldsymbol{F}_f$ is the frictional force.



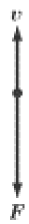
In cases like the diagram above, it's very easy to deal with the force of tension by treating the situation just as if there were somebody behind the box pushing on it. We'll find the force of tension coming up quite a bit in the chapter on special problems in mechanics, particularly when we deal with pulleys.

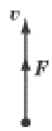
Key Formulas

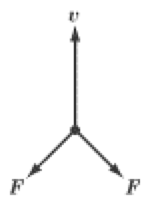
Newton's Second Law	$\mathbf{F} = m\mathbf{a}$
Formula for Force of Kinetic Friction	$F_{\text{kinetic}} = \mu_k N$
Formula for Force of Maximum Static Friction	$F_{\text{static, max}} = \mu_s N$

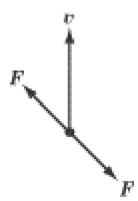
Practice Questions

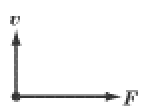
1. Each of the figures below shows a particle moving with velocity ***v***, and with one or two forces of magnitude *F* acting upon it. In which of the figures will ***v*** remain constant?

   (A) 

   (B) 

   (C) 

   (D) 

   (E) 

2. In which of the following examples is a net force of zero acting on the object in question?

   I. A car drives around a circular racetrack at a constant speed
   II. A person pushes on a door to hold it shut
   III. A ball, rolling across a grassy field, slowly comes to a stop

   (A) I only

   (B) II only

   (C) III only

   (D) I and II only

   (E) I and III only

3. A force ***F*** is acting on an object of mass *m* to give it an acceleration of ***a***. If *m* is halved and ***F*** is quadrupled, what happens to ***a***?

   (A) It is divided by eight

   (B) It is divided by two

   (C) It remains unchanged

   (D) It is multiplied by two

   (E) It is multiplied by eight

4. A force F_1 pushes on an object of mass 10 kg with a force of 5 N to the right. A force F_2 pushes on the same object with a force of 15 N to the left. What is the acceleration of the object?

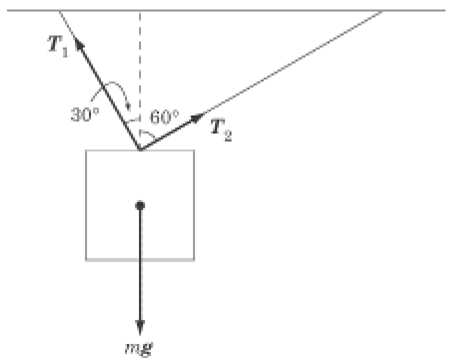
(A) 0.3 m/s^2 to the left

(B) 0.5 m/s^2 to the left

(C) 1 m/s^2 to the left

(D) 1.5 m/s^2 to the left

(E) 10 m/s^2 to the left



5. In the figure above, a block is suspended from two ropes, so that it hangs motionless in the air. If the magnitude of T_2 is 10.0 N, what is the magnitude of T_1? Note that sin 30 = cos 60 = 0.500, and sin 60 = cos 30 = 0.866.

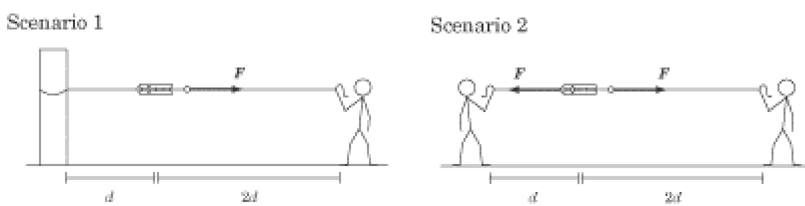
(A) 0.433 N

(B) 0.500 N

(C) 0.866 N

(D) 10.0 N

(E) 17.3 N



6. In scenario 1, a person pulls with a force ***F*** on a string of length 2*d* that is connected to a spring scale. The other end of the spring scale is connected to a post by a string of length *d*. In scenario 2, the person pulls on the string of length 2*d* with a force of ***F***, and a second person stands where the post was in scenario 1, and also pulls with a force of ***F***. If the spring scale reads 50 N in scenario 1, what does the spring scale read in scenario 2?

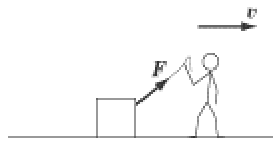
(A) 50 N

(B) 67 N

(C) 100 N

(D) 133 N

(E) 150 N



7. In the figure above, a person is dragging a box attached to a string along the ground. Both the person and the box are moving to the right with a constant velocity, *v*. What horizontal forces are acting on the person?
   (A) The tension force in the string is pulling the person to the left
   (B) The tension force in the string is pulling the person to the left, and the Earth is pushing the person to the right
   (C) The tension force in the string is pulling the person to the left, and the Earth is pushing the person to the left
   (D) The tension force in the string is pushing the person to the right, and the Earth is pushing the person to the right
   (E) The tension force in the string is pushing the person to the right, and the Earth is pushing the person to the left

8. What is the weight of a man whose mass is 80 kg?
   (A) 8.1 N
   (B) 70.2 N
   (C) 80 N
   (D) 89.8 N
   (E) 784 N

9. A 50 kg crate rests on the floor. The coefficient of static friction is 0.5. The force parallel to the floor needed to move the crate is most nearly:
   (A) 25 N
   (B) 50 N
   (C) 125 N
   (D) 250 N
   (E) 500 N

10. A person is pushing an object of mass *m* along the ground with a force ***F***. The coefficient of kinetic friction between the object and the ground is μ_k. The object is accelerating, but then the person stops pushing and the object slides to a halt. The person then starts pushing on the object again with a force ***F***, but the object doesn't budge. The maximum coefficient of static friction between the object and the ground is μ_s. Which of the following statements is true?
    (A) $F > \frac{\mu_k}{\mu_s}$
    (B) $\mu_k > \mu_s$
    (C) $\mu_k mg < F \leq \mu_s mg$
    (D) $\mu_s mg = F$
    (E) The scenario described is physically impossible

Work, Energy, and Power

THERE ARE A NUMBER OF TECHNICAL terms in physics that have a nontechnical equivalent in ordinary usage. An example we saw in the previous chapter is force. We can talk about force in conversation without meaning a push or a pull that changes the velocity of an object, but it's easy to see that that technical definition has something in common with the ordinary use of the word *force*. The same is true with *work*, *energy*, and *power*. All three of these words have familiar connotations in ordinary speech, but in physics they take on a technical meaning. As with force, the ordinary meaning of these words provides us with some hint as to their meaning in physics. However, we shouldn't rely too heavily on our intuition, since, as we shall see, there are some significant divergences from what common sense tells us.

The related phenomena of work, energy, and power find their way into a good number of questions on SAT II Physics. And energy, like force, finds its way into almost every aspect of physics, so a mastery of this subject matter is very important. The **conservation of energy** is one of the most important laws of physics, and conveniently serves as a tool to sort out many a head-splitting physics problem.

Work

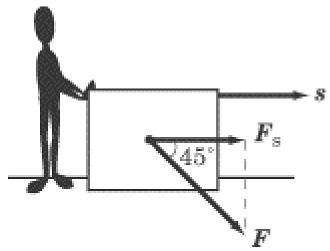
When we are told that a person pushes on an object with a certain force, we only know how hard the person pushes: we don't know what the pushing accomplishes. **Work**, W, a scalar quantity that measures the product of the force exerted on an object and the resulting displacement of that object, is a measure of what an applied force accomplishes. The harder you push an object, and the farther that object travels, the more work you have done. In general, we say that work is done *by* a force, or *by* the object or person exerting the force, *on* the object on which the force is acting. Most simply, work is the product of force times displacement. However, as you may have remarked, both force and displacement are vector quantities, and so the direction of these vectors comes into play when calculating the work done by a given force. Work is measured in units of **joules** (J), where 1 J = 1 N · m = 1 kg · m^2/s^2.

Work When Force and Displacement Are Parallel

When the force exerted on an object is in the same direction as the displacement of the object, calculating work is a simple matter of multiplication. Suppose you exert a force of 10 N on a box in the northward direction, and the box moves 5 m to the north. The work you have done on the box is $10 \text{ N} \times 5 \text{ m} = 50$ N · m = 50 J. If force and displacement are parallel to one another, then the work done by a force is simply the product of the magnitude of the force and the magnitude of the displacement.

Work When Force and Displacement Are Not Parallel

Unfortunately, matters aren't quite as simple as scalar multiplication when the force and displacement vectors aren't parallel. In such a case, we define work as the product of the displacement of a body and the component of the force in the direction of that displacement. For instance, suppose you push a box with a force F along the floor for a distance s, but rather than pushing it directly forward, you push on it at a downward angle of 45°. The work you do on the box is not equal to $F \times s$, the magnitude of the force times the magnitude of the displacement. Rather, it is equal to $F_s \times s$, the magnitude of the force exerted in the direction of the displacement times the magnitude of the displacement.



Some simple trigonometry shows us that $F_s = F\cos\theta$, where θ is the angle between the ***F*** vector and the ***s*** vector. With this in mind, we can express a general formula for the work done by a force, which applies to all cases:

$$W = \boldsymbol{F_s} \times \boldsymbol{s} = Fs\cos\theta$$

This formula also applies to the cases where ***F*** and ***s*** are parallel, since in those cases, $\theta = 0$, and $\cos\theta = 1$, so $W = F_s$.

Dot Product

What the formula above amounts to is that work is the dot product of the force vector and the displacement vector. As we recall, the dot product of two vectors is the product of the magnitudes of the two vectors multiplied by the cosine of the angle between the two vectors. So the most general vector definition of work is:

$$W = \boldsymbol{F} \cdot \boldsymbol{s} = Fs\cos\theta$$

Review

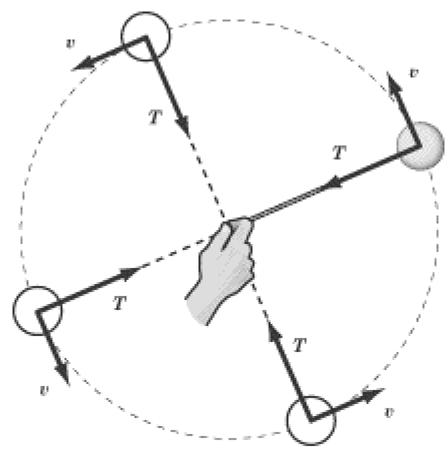
The concept of work is actually quite straightforward, as you'll see with a little practice. You just need to bear a few simple points in mind:

- If force and displacement are both in the same direction, the work done is the product of the magnitudes of force and displacement.
- If force and displacement are at an angle to one another, you need to calculate the component of the force that points in the direction of the displacement, or the component of the displacement that points in the direction of the force. The work done is the product of the one vector and the component of the other vector.
- If force and displacement are perpendicular, no work is done.

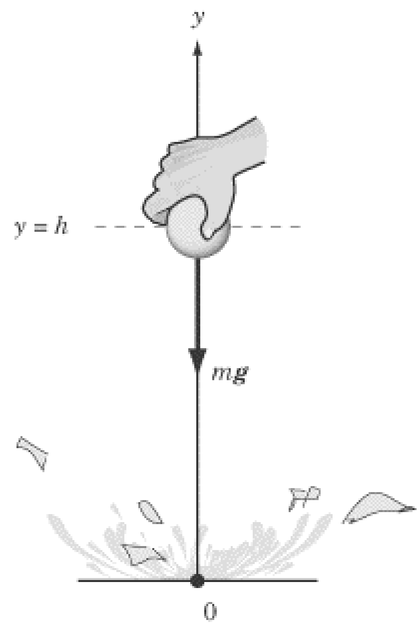
Because of the way work is defined in physics, there are a number of cases that go against our everyday intuition. Work is not done whenever a force is exerted, and there are certain cases in which we might think that a great deal of work is being done, but in fact no work is done at all. Let's look at some examples that might be tested on SAT II Physics:

- You do work on a 10 kg mass when you lift it off the ground, but you do no work to hold the same mass stationary in the air. As you strain to hold the mass in the air, you are actually making sure that it is not displaced. Consequently, the work you do to hold it is zero.

- Displacement is a vector quantity that is not the same thing as distance traveled. For instance, if a weightlifter raises a dumbbell 1 m, then lowers it to its original position, the weightlifter has not done any work on the dumbbell.
- When a force is perpendicular to the direction of an object's motion, this force does not work on the object. For example, say you swing a tethered ball in a circle overhead, as in the diagram below. The tension force, ***T***, is always perpendicular to the velocity, ***v***, of the ball, and so the rope does not work on the ball.



EXAMPLE



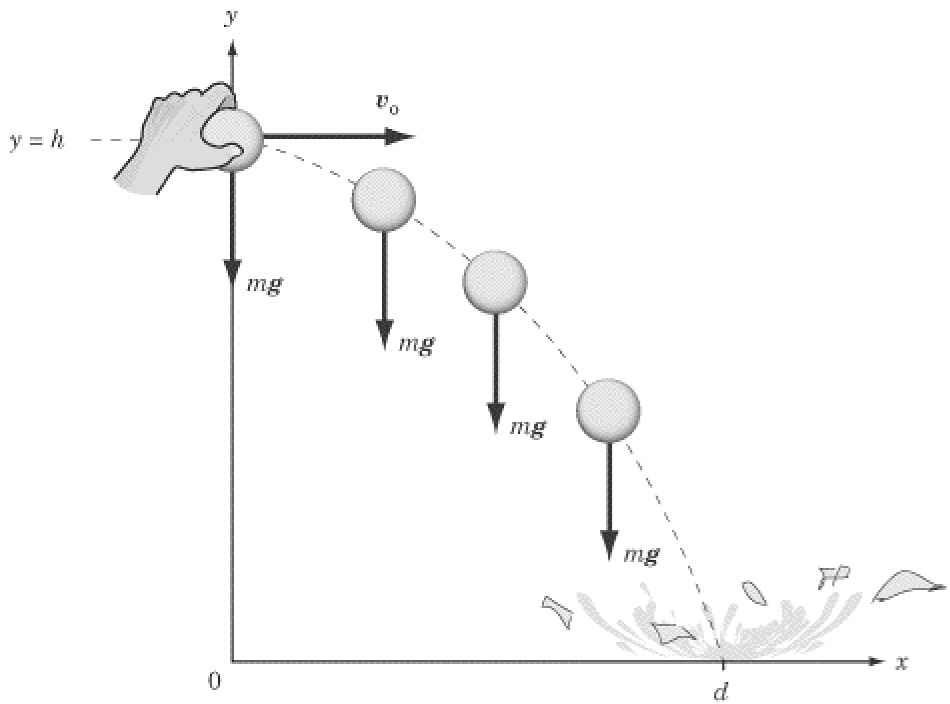
A water balloon of mass *m* is dropped from a height *h*. What is the work done on the balloon by gravity? How much work is done by gravity if the balloon is thrown horizontally from a height *h* with an initial velocity of v_0 ?

WHAT IS THE WORK DONE ON THE BALLOON BY GRAVITY?

Since the gravitational force of $-m\mathbf{g}$ is in the same direction as the water balloon's displacement, $-\mathbf{h}$, the work done by the gravitational force on the ball is the force times the displacement, or $W = mgh$, where $\mathbf{g} = -9.8$ m/s².

HOW MUCH WORK IS DONE BY GRAVITY IF THE BALLOON IS THROWN HORIZONTALLY FROM A HEIGHT *H* WITH AN INITIAL VELOCITY OF *V*0?

The gravitational force exerted on the balloon is still $-m\mathbf{g}$, but the displacement is different. The balloon has a displacement of $-\mathbf{h}$ in the y direction and d (see the figure below) in the x direction. But, as we recall, the work done on the balloon by gravity is not simply the product of the magnitudes of the force and the displacement. We have to multiply the force by the component of the displacement that is parallel to the force. The force is directed downward, and the component of the displacement that is directed downward is $-h$. As a result, we find that the work done by gravity is mgh, just as before.



The work done by the force of gravity is the same if the object falls straight down or if it makes a wide parabola and lands 100 m to the east. This is because the force of gravity does not work when an object is transported horizontally, because the force of gravity is perpendicular to the horizontal component of displacement.

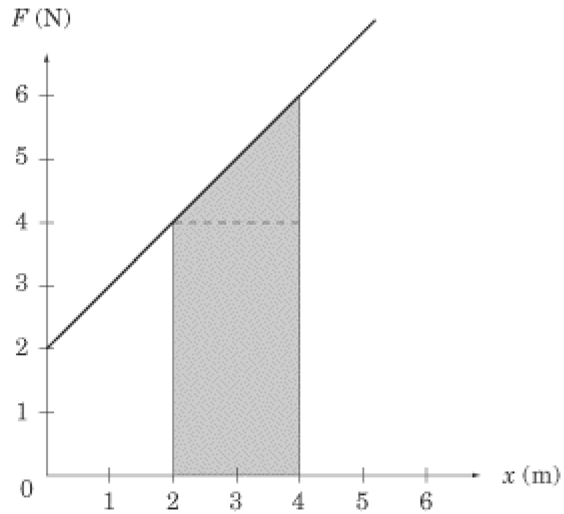
Work Problems with Graphs

There's a good chance SAT II Physics may test your understanding of work by asking you to interpret a graph. This graph will most likely be a force vs. position graph, though there's a chance it may be a graph of $F\cos\theta$ vs. position. Don't let the appearance of trigonometry scare you: the principle of reading graphs is the same in both cases. In the latter case, you'll be dealing with a graphic representation of a force that isn't acting parallel to the displacement, but the graph will have already taken this into account. Bottom line: all graphs dealing with work will operate according to the same easy principles. The most important thing that you need to remember about these graphs is:

The work done in a force vs. displacement graph is equal to the area between the graph and the x-axis during the same interval.

If you recall your kinematics graphs, this is exactly what you would do to read velocity on an acceleration vs. time graph, or displacement on a velocity vs. time graph. In fact, whenever you want a quantity that is the product of the quantity measured by the *y*-axis and the quantity measured by the *x*-axis, you can simply calculate the area between the graph and the *x*-axis.

EXAMPLE



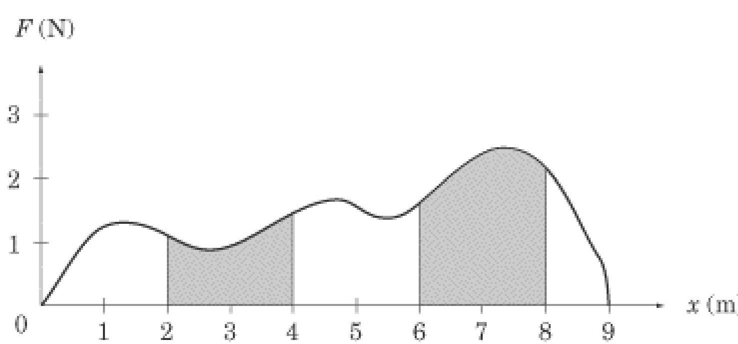
The graph above plots the force exerted on a box against the displacement of the box. What is the work done by the force in moving the box from $x = 2$ to $x = 4$?

The work done on the box is equal to the area of the shaded region in the figure above, or the area of a rectangle of width 2 and height 4 plus the area of a right triangle of base 2 and height 2. Determining the amount of work done is simply a matter of calculating the area of the rectangle and the area of the triangle, and adding these two areas together:

$$2 \cdot 4 + \frac{1}{2} \cdot 2 \cdot 2 = 10 \text{ J}$$

Curved Force vs. Position Graphs

If SAT II Physics throws you a curved force vs. position graph, don't panic. You won't be asked to calculate the work done, because you can't do that without using calculus. Most likely, you'll be asked to estimate the area beneath the curve for two intervals, and to select the interval in which the most, or least, work was done. In the figure below, more work was done between $x = 6$ and $x = 8$ than between $x = 2$ and $x = 4$, because the area between the graph and the x-axis is larger for the interval between $x = 6$ and $x = 8$.



Energy

Energy is one of the central concepts of physics, and one of the most difficult to define. One of the reasons we have such a hard time defining it is because it appears in so many different forms. There is the **kinetic** and **potential energy** of kinematic motion, the **thermal energy** of heat reactions, the chemical energy of your discman batteries, the **mechanical energy** of a machine, the elastic energy that helps you launch rubber bands, the electrical energy that keeps most appliances on this planet running, and even mass energy, the strange phenomenon that Einstein discovered and that has been put to such devastating effect in the atomic bomb. This is only a cursory list: energy takes on an even wider variety of forms.

How is it that an electric jolt, a loud noise, and a brick falling to the ground can all be treated using the same concept? Well, one way of defining energy is as a capacity to do work: any object or phenomenon that is capable of doing work contains and expends a certain amount of energy. Because anything that can exert a force or have a force exerted on it can do work, we find energy popping up wherever there are forces.

Energy, like work, is measured in joules (J). In fact, work is a measure of the transfer of energy. However, there are forms of energy that do not involve work. For instance, a box suspended from a string is doing no work, but it has **gravitational potential energy** that will turn into work as soon as the string is cut. We will look at some of the many forms of energy shortly. First, let's examine the important law of conservation of energy.

Conservation of Energy

As the name suggests, the law of conservation of energy tells us that the energy in the universe is constant. Energy cannot be made or destroyed, only changed from one form to another form. Energy can also be transferred via a force, or as heat. For instance, let's return to the example mentioned earlier of the box hanging by a string. As it hangs motionless, it has gravitational potential energy, a kind of latent energy. When we cut the string, that energy is converted into **kinetic energy**, or work, as the force of gravity acts to pull the box downward. When the box hits the ground, that kinetic energy does not simply disappear. Rather, it is converted into sound and heat energy: the box makes a loud thud and the impact between the ground and the box generates a bit of heat.

This law applies to any closed system. A closed system is a system where no energy leaves the system and goes into the outside world, and no energy from the outside world enters the system. It is virtually impossible to create a truly closed system on Earth, since energy is almost always dissipated through friction, heat, or sound, but we can create close approximations. Objects sliding over ice or air hockey tables move with a minimal amount of friction, so the energy in these systems remains nearly constant. Problems on SAT II Physics that quiz you on the conservation of energy will almost always deal with frictionless surfaces, since the law of conservation of energy applies only to closed systems.

The law of conservation of energy is important for a number of reasons, one of the most fundamental being that it is so general: it applies to the whole universe and extends across all time. For the purposes of SAT II Physics, it helps you solve a number of problems that would be very difficult otherwise. For example, you can often determine an object's velocity quite easily by using this law, while it might have been very difficult or even impossible using only kinematic equations. We will see this law at work later in this chapter, and again when we discuss elastic and inelastic collisions in the chapter on linear momentum.

Forms of Energy

Though energy is always measured in joules, and though it can always be defined as a capacity to do work, energy manifests itself in a variety of different forms. These various forms pop up all over SAT II Physics, and we will look at some additional forms of energy when we discuss electromagnetism, relativity, and a number of other specialized topics. For now, we will focus on the kinds of energy you'll find in mechanics problems.

Kinetic Energy

Kinetic energy is the energy a body in motion has by virtue of its motion. We define energy as the capacity to do work, and a body in motion is able to use its motion to do work. For instance, a cue ball on a pool table can use its motion to do work on the eight ball. When the cue ball strikes the eight ball, the cue ball comes to a stop and the eight ball starts moving. This occurs because the cue ball's kinetic energy has been transferred to the eight ball.

There are many types of kinetic energy, including vibrational, translational, and rotational. **Translational kinetic energy**, the main type, is the energy of a particle moving in space and is defined in terms of the particle's mass, m, and velocity, v:

$$KE = \frac{1}{2}mv^2$$

For instance, a cue ball of mass 0.5 kg moving at a velocity of 2 m/s has a kinetic energy of $\frac{1}{2}(0.5 \text{ kg})(2 \text{ m/s})^2 = 1$ J.

The Work-Energy Theorem

If you recall, work is a measure of the transfer of energy. An object that has a certain amount of work done on it has that amount of energy transferred to it. This energy moves the object over a certain distance with a certain force; in other words, it is kinetic energy. This handy little fact is expressed in the **work-energy theorem**, which states that the net work done on an object is equal to the object's change in kinetic energy:

$$W = \Delta KE$$

For example, say you apply a force to a particle, causing it to accelerate. This force does positive work on the particle and increases its kinetic energy. Conversely, say you apply a force to decelerate a particle. This force does negative work on the particle and decreases its kinetic energy. If you know the forces acting on an object, the work-energy theorem provides a convenient way to calculate the velocity of a particle.

EXAMPLE

> A hockey puck of mass 1 kg slides across the ice with an initial velocity of 10 m/s. There is a 1 N force of friction acting against the puck. What is the puck's velocity after it has glided 32 m along the ice?

If we know the puck's kinetic energy after it has glided 32 m, we can calculate its velocity. To determine its kinetic energy at that point, we need to know its initial kinetic energy, and how much that kinetic energy changes as the puck glides across the ice.

First, let's determine the initial kinetic energy of the puck. We know the puck's initial mass and initial velocity, so we just need to plug these numbers into the equation for kinetic energy:

$$KE = \frac{1}{2}mv^2 = \frac{1}{2}(1\text{ kg})(10\text{ m/s})^2 = 50\text{ J}$$

The friction between the puck and the ice decelerates the puck. The amount of work the ice does on the puck, which is the product of the force of friction and the puck's displacement, is negative.

$$W = \boldsymbol{F} \cdot \boldsymbol{s} = (-1\text{ N})(32\text{ m}) = 32\text{ J}$$

The work done on the puck decreases its kinetic energy, so after it has glided 32 m, the kinetic energy of the puck is $50 - 32 = 18$ J. Now that we know the final kinetic energy of the puck, we can calculate its final velocity by once more plugging numbers into the formula for kinetic energy:

$$\begin{aligned} KE &= \frac{1}{2}mv^2 \\ 18\text{ J} &= \frac{1}{2}(1\text{ kg})v^2 \\ v^2 &= (36\text{ m/s})^2 \\ v &= 6\text{ m/s} \end{aligned}$$

We could also have solved this problem using Newton's Second Law and some kinematics, but the work-energy theorem gives us a quicker route to the same answer.

Potential Energy

As we said before, work is the process of energy transfer. In the example above, the kinetic energy of the puck was transferred into the heat and sound caused by friction. There are a great number of objects, though, that spend most of their time neither doing work nor having work done on them. This book in your hand, for instance, is not doing any work right now, but the second you drop it—whoops!—the force of gravity does some work on it, generating kinetic energy. Now pick up the book and let's continue.

Potential energy, U, is a measure of an object's unrealized potential to have work done on it, and is associated with that object's position in space, or its configuration in relation to other objects. Any work done on an object converts its potential energy into kinetic energy, so the net work done on a given object is equal to the negative change in its potential energy:

$$W = -\Delta U$$

Be very respectful of the minus sign in this equation. It may be tempting to think that the work done on an object increases its potential energy, but the opposite is true. Work converts potential energy into other forms of energy, usually kinetic energy. Remove the minus sign from the equation above, and you are in direct violation of the law of conservation of energy!

There are many forms of potential energy, each of which is associated with a different type of force. SAT II Physics usually confines itself to gravitational potential energy and the potential energy of a compressed spring. We will review gravitational potential energy in this section, and the potential energy of a spring in the next chapter.

Gravitational Potential Energy

Gravitational potential energy registers the potential for work done on an object by the force of gravity. For example, say that you lift a water balloon to height h above the ground. The work done by the force of gravity as you lift the water balloon is the force of gravity, $-mg$, times the water balloon's displacement, h. So the work done by the force of gravity is $W = -mgh$. Note that there is a negative amount of work done, since the water balloon is being lifted upward, in the opposite direction of the force of gravity.

By doing $-mgh$ joules of work on the water balloon, you have increased its gravitational potential energy by mgh joules (recall the equation $W = -\Delta U$). In other words, you have increased its potential to accelerate downward and cause a huge splash. Because the force of gravity has the potential to do mgh joules of work on the water balloon at height h, we say that the water balloon has mgh joules of gravitational potential energy.

$$U_g = mgh$$

For instance, a 50 kg mass held at a height of 4 m from the ground has a gravitational potential energy of:

$$U_g = mgh = (50\text{ kg})(9.8\text{ m/s}^2)(4\text{ m}) = 1960\text{ J}$$

The most important thing to remember is that *the higher an object is off the ground, the greater its gravitational potential energy.*

Mechanical Energy

We now have equations relating work to both kinetic and potential energy:

$$W = \Delta KE$$
$$W = -\Delta U$$

Combining these two equations gives us this important result:

$$\Delta KE + \Delta U = 0$$

Or, alternatively,

$$\Delta KE = -\Delta U$$

As the kinetic energy of a system increases, its potential energy decreases by the same amount, and vice versa. As a result, the sum of the kinetic energy and the potential energy in a system is constant. We define this constant as E, the mechanical energy of the system:

$$KE + U = E$$

This law, the conservation of mechanical energy, is one form of the more general law of conservation of energy, and it's a handy tool for solving problems regarding projectiles, pulleys, springs, and inclined planes. However, mechanical energy is *not* conserved in problems involving frictional forces. When friction is involved, a good deal of the energy in the system is dissipated as heat and sound. The conservation of mechanical energy only applies to closed systems.

EXAMPLE 1

A student drops an object of mass 10 kg from a height of 5 m. What is the velocity of the object when it hits the ground? Assume, for the purpose of this question, that $\boldsymbol{g}$ = –10 m/s^2.

Before the object is released, it has a certain amount of gravitational potential energy, but no kinetic energy. When it hits the ground, it has no gravitational potential energy, since $h = 0$, but it has a certain amount of kinetic energy. The mechanical energy, E, of the object remains constant, however. That means that the potential energy of the object before it is released is equal to the kinetic energy of the object when it hits the ground.

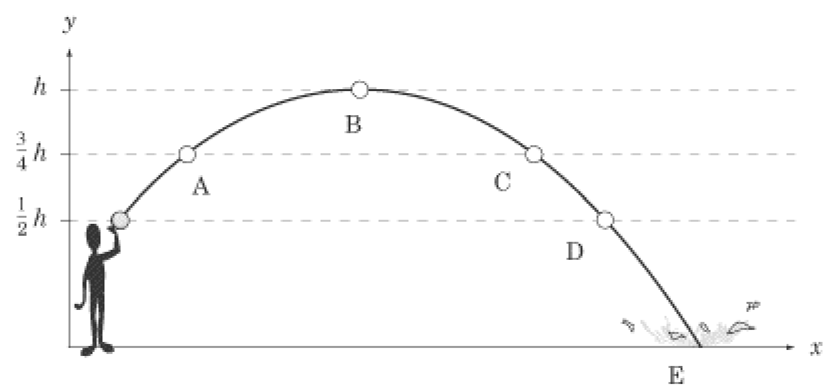
When the object is dropped, it has a gravitational potential energy of:

$$mgh = (10 \text{ kg})(-10 \text{ m/s}^2)(-5 \text{ m}) = 500 \text{ J}$$

By the time it hits the ground, all this potential energy will have been converted to kinetic energy. Now we just need to solve for v:

$$\begin{aligned} \frac{1}{2}mv^2 &= 500 \text{ J} \\ v^2 &= \frac{2(500 \text{ J})}{10 \text{ kg}} \\ &= (100 \text{ m/s})^2 \\ v &= 10 \text{ m/s} \end{aligned}$$

EXAMPLE 2



Consider the above diagram of the trajectory of a thrown tomato:

1. At what point is the potential energy greatest?
2. At what point is the kinetic energy the least?
3. At what point is the kinetic energy greatest?
4. At what point is the kinetic energy decreasing and the potential energy increasing?
5. At what point are the kinetic energy and the potential energy equal to the values at position A?

The answer to question 1 is point B. At the top of the tomato’s trajectory, the tomato is the greatest distance above the ground and hence has the greatest potential energy.

The answer to question 2 is point B. At the top of the tomato’s trajectory, the tomato has the smallest velocity, since the y-component of the velocity is zero, and hence the least kinetic energy. Additionally, since mechanical energy is conserved in projectile motion, we know that the point where the potential energy is the greatest corresponds to the point where the kinetic energy is smallest.

The answer to question 3 is point E. At the bottom of its trajectory, the tomato has the greatest velocity and thus the greatest kinetic energy.

The answer to question 4 is point A. At this point, the velocity is decreasing in magnitude and the tomato is getting higher in the air. Thus, the kinetic energy is decreasing and the potential energy is increasing.

The answer to question 5 is point C. From our study of kinematics, we know that the speed of a projectile is equal at the same height in the projectile's ascent and descent. Therefore, the tomato has the same kinetic energy at points A and C. Additionally, since the tomato has the same height at these points, its potential energy is the same at points A and C.

Keep this example in mind when you take SAT II Physics, because it is likely that a similar question will appear on the test.

Thermal Energy

There are many cases where the energy in a system seems simply to have disappeared. Usually, this is because that energy has been turned into sound and heat. For instance, a coin sliding across a table slows down and comes to a halt, but in doing so, it produces the sound energy of the coin scraping along the table and the heat energy of friction. Rub your hands together briskly and you will feel that friction causes heat.

We will discuss thermal energy, or heat, in greater detail in Chapter 9, but it's worth noting here that it is the most common form of energy produced in energy transformations. It's hard to think of an energy transformation where no heat is produced. Take these examples:

- Friction acts everywhere, and friction produces heat.
- Electric energy produces heat: a light bulb produces far more heat than it does light.
- When people talk about burning calories, they mean it quite literally: exercise is a way of converting food energy into heat.
- Sounds fade to silence because the sound energy is gradually converted into the heat of the vibrating air molecules. In other words, if you shout very loudly, you make the air around you warmer!

Power

Power is an important physical quantity that frequently, though not always, appears on SAT II Physics. Mechanical systems, such as engines, are not limited by the amount of work they can do, but rather by the rate at which they can perform the work. Power, P, is defined as the rate at which work is done, or the rate at which energy is transformed. The formula for average power is:

$$\frac{\Delta W}{\Delta t} \text{ or } \frac{\Delta E}{\Delta t}$$

Power is measured in units of watts (W), where 1 W = 1 J/s.

EXAMPLE

> A piano mover pushes on a piano with a force of 100 N, moving it 9 m in 12 s. With how much power does the piano mover push?

Power is a measure of the amount of work done in a given time period. First we need to calculate how much work the piano mover does, and then we divide that quantity by the amount of time the work takes.

$$W = Fs = (100\text{ N})(9\text{ m}) = 900\text{ J}$$
$$P = \frac{\Delta W}{\Delta t} = \frac{900\text{ J}}{12\text{ s}} = 75\text{ W}$$

Be careful not to confuse the symbol for watts, W, with the symbol for work, *W*.

Instantaneous Power

Sometimes we may want to know the instantaneous power of an engine or person, the amount of power output by that person at any given instant. In such cases, there is no value for Δt to draw upon. However, when a steady force is applied to an object, the change in the amount of work done on the object is the product of the force and the change in that object's displacement. Bearing this in mind, we can express power in terms of force and velocity:

$$\begin{aligned} P &= \frac{\Delta W}{\Delta t} \\ &= \boldsymbol{F}\frac{\Delta \boldsymbol{s}}{\Delta t} \\ &= \boldsymbol{F} \cdot \boldsymbol{v} \end{aligned}$$

Key Formulas

Work	$W = \boldsymbol{F} \cdot \boldsymbol{s} = Fs\cos\theta$
Work Done by Gravity	$W = mgh$
Kinetic Energy	$KE = \frac{1}{2}mv^2$
Work-Energy Theorem	$W = \Delta KE$
Potential Energy	$W = -\Delta U$
Gravitational Potential Energy	$U_g = mgh$
Mechanical Energy	$E = KE + U$
Average Power	$\frac{\Delta W}{\Delta t}$ or $\frac{\Delta E}{\Delta t}$
Instantaneous Power	$P = \boldsymbol{F} \cdot \boldsymbol{v}$

Practice Questions

1. How much work does a person do in pushing a box with a force of 10 N over a distance of 4.0 m in the direction of the force?

   (A) 0.4 J

   (B) 4.0 J

   (C) 40 J

   (D) 400 J

   (E) 4000 J

2. A person pushes a 10 kg box at a constant velocity over a distance of 4 m. The coefficient of kinetic friction between the box and the floor is 0.3. How much work does the person do in pushing the box?

   (A) 12 J

   (B) 40 J

   (C) 75 J

   (D) 120 J

   (E) 400 J

3. How much work does the force of gravity do in pulling a 10 kg box down a 30Âº inclined plane of length 8.0 m? Note that sin 30 = cos 60 = 0.500 and cos 30 = sin 60 = 0.866.

   (A) 40 J

   (B) 69 J

   (C) 400 J

   (D) 690 J

   (E) 800 J

4. How much work does a person do in pushing a box with a force of 20 N over a distance of 8.0 m in the direction of the force?

   (A) 1.6 J

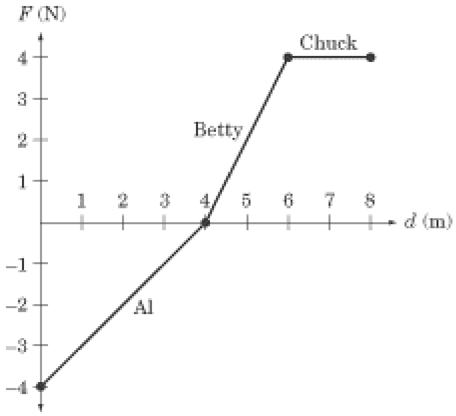
   (B) 16 J

   (C) 160 J

   (D) 1600 J

   (E) 16000 J

5. The figure below is a force vs. displacement graph, showing the amount of force applied to an object by three different people. Al applies force to the object for the first 4 m of its displacement, Betty applies force from the 4 m point to the 6 m point, and Chuck applies force from the 6 m point to the 8 m point. Which of the three does the most work on the object?



(A) Al

(B) Betty

(C) Chuck

(D) Al and Chuck do the same amount of work

(E) Betty and Chuck do the same amount of work

6. When a car's speed doubles, what happens to its kinetic energy?

(A) It is quartered

(B) It is halved

(C) It is unchanged

(D) It is doubled

(E) It is quadrupled

7. A worker does 500 J of work on a 10 kg box. If the box transfers 375 J of heat to the floor through the friction between the box and the floor, what is the velocity of the box after the work has been done on it?

(A) 5 m/s

(B) 10 m/s

(C) 12.5 m/s

(D) 50 m/s

(E) 100 m/s

8. A person on the street wants to throw an 8 kg book up to a person leaning out of a window 5 m above street level. With what velocity must the person throw the book so that it reaches the person in the window?

(A) 5 m/s

(B) 8 m/s

(C) 10 m/s

(D) 40 m/s

(E) 50 m/s

Questions 9 and 10 refer to a forklift lifting a crate of mass 100 kg at a constant velocity to a height of 8 m over a time of 4 s. The forklift then holds the crate in place for 20 s.

9. How much power does the forklift exert in lifting the crate?

(A) 0 W

(B) 2.0×10^3 W

(C) 3.2×10^3 W

(D) 2.0×10^4 W

(E) 3.2×10^4 W

10. How much power does the forklift exert in holding the crate in place?

(A) 0 W

(B) 400 W

(C) 1.6×10^3 W

(D) 4.0×10^3 W

(E) 1.6×10^4 W

Special Problems in Mechanics

THE “SPECIAL PROBLEMS” WE WILL address in this chapter deal with four common mechanical systems: pulleys, inclined planes, springs, and pendulums. These systems pop up on many mechanics problems on SAT II Physics, and it will save you time and points if you familiarize yourself with their quirks. These systems obey the same mechanical rules as the rest of the world, and we will only introduce one principle (Hooke’s Law) that hasn’t been covered in the previous three chapters. However, there are a number of problem-solving techniques that are particular to these sorts of problems, and mastering them will help you get through these problems quickly and easily.

The Three-Step Approach to Problem Solving

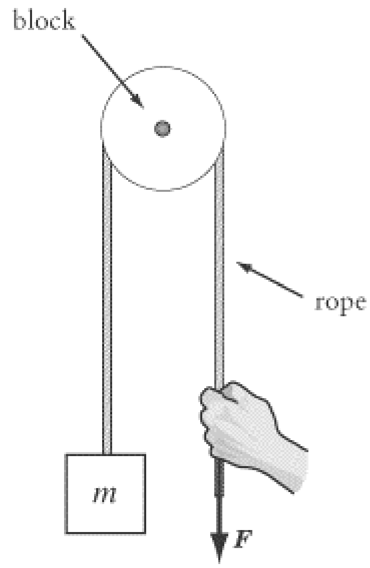
The systems we will look at in this chapter won’t test your knowledge of obscure formulas so much as your problem-solving abilities. The actual physics at work on these systems is generally quite simple—it rarely extends beyond Newton’s three laws and a basic understanding of work and energy—but you’ll need to apply this simple physics in imaginative ways.

There are three general steps you can take when approaching any problem in mechanics. Often the problems are simple enough that these steps are unnecessary. However, with the special problems we will tackle in this chapter, following these steps carefully may save you many times over on SAT II Physics. The three steps are:

1. **Ask yourself how the system will move:** Before you start writing down equations and looking at answer choices, you should develop an intuitive sense of what you’re looking at. In what direction will the objects in the system move? Will they move at all? Once you know what you’re dealing with, you’ll have an easier time figuring out how to approach the problem.
2. **Choose a coordinate system:** Most systems will only move in one dimension: up and down, left and right, or on an angle in the case of inclined planes. Choose a coordinate system where one direction is negative, the other direction is positive, and, if necessary, choose an origin point that you label 0. Remember: no coordinate system is right or wrong in itself, some are just more convenient than others. The important thing is to be strictly consistent once you’ve chosen a coordinate system, and to be mindful of those subtle but crucial minus signs!
3. **Draw free-body diagrams:** Most students find mechanics easier than electromagnetism for the simple reason that mechanics problems are easy to visualize. Free-body diagrams allow you to make the most of this advantage. Make sure you’ve accounted for all the forces acting on all the bodies in the system. Make ample use of Newton’s Third Law, and remember that for systems at rest or at a constant velocity, the net force acting on everybody in the system must be zero.

Students too often think that physics problem solving is just a matter of plugging the right numbers into the right equations. The truth is, physics problem solving is more a matter of determining what those right numbers and right equations are. These three steps should help you do just that. Let’s look at some mechanical systems.

Pulleys

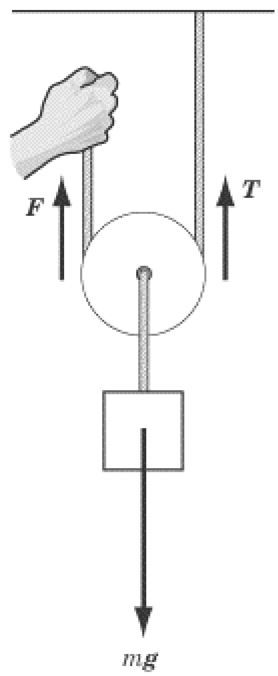


Pulleys are simple machines that consist of a rope that slides around a disk, called a block. Their main function is to change the direction of the tension force in a rope. The pulley systems that appear on SAT II Physics almost always consist of idealized, mass less and frictionless pulleys, and idealized ropes that are mass less and that don't stretch. These somewhat unrealistic parameters mean that:

1. The rope slides without any resistance over the pulley, so that the pulley changes the direction of the tension force without changing its magnitude.
2. You can apply the law of conservation of energy to the system without worrying about the energy of the rope and pulley.
3. You don't have to factor in the mass of the pulley or rope when calculating the effect of a force exerted on an object attached to a pulley system.

The one exception to this rule is the occasional problem you might find regarding the torque applied to a pulley block. In such a problem, you will have to take the pulley's mass into account. We'll deal with this special case in Chapter 7, when we look at torque.

The Purpose of Pulleys

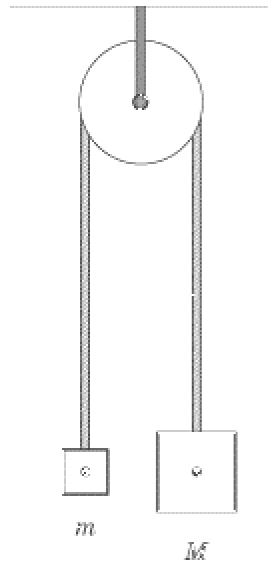


We use pulleys to lift objects because they reduce the amount of force we need to exert. For example, say that you are applying force $\boldsymbol{F}$ to the mass in the figure above. How does $\boldsymbol{F}$ compare to the force you would have to exert in the absence of a pulley?

To lift mass m at a constant velocity without a pulley, you would have to apply a force equal to the mass's weight, or a force of mg upward. Using a pulley, the mass must still be lifted with a force of mg upward, but this force is distributed between the tension of the rope attached to the ceiling, $\boldsymbol{T}$, and the tension of the rope gripped in your hand, $\boldsymbol{F}$.

Because there are two ropes pulling the block, and hence the mass, upward, there are two equal upward forces, $\boldsymbol{F}$ and $\boldsymbol{T}$. We know that the sum of these forces is equal to the gravitational force pulling the mass down, so $\boldsymbol{F} + \boldsymbol{T} = 2\boldsymbol{F} = mg$ or $\boldsymbol{F} = mg/2$. Therefore, you need to pull with only one half the force you would have to use to lift mass m if there were no pulley.

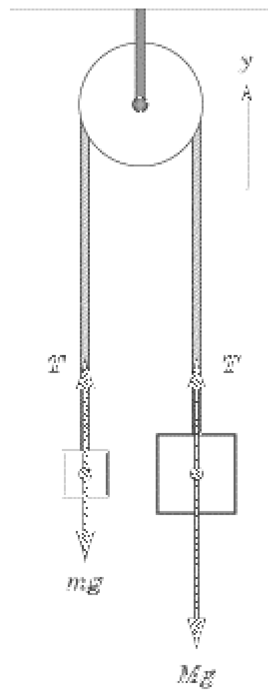
Standard Pulley Problem



The figure above represents a pulley system where masses m and M are connected by a rope over a mass less and frictionless pulley. Note that $M > m$ and both masses are at the same height above the ground. The system is initially held at rest, and is then released. We will learn to calculate the acceleration of the masses, the velocity of mass m when it moves a distance h, and the work done by the tension force on mass m as it moves a distance h.

Before we start calculating values for acceleration, velocity, and work, let's go through the three steps for problem solving:

1. **Ask yourself how the system will move:** From experience, we know that the heavy mass, M, will fall, lifting the smaller mass, m. Because the masses are connected, we know that the velocity of mass m is equal in magnitude to the velocity of mass M, but opposite in direction. Likewise, the acceleration of mass m is equal in magnitude to the acceleration of mass M, but opposite in direction.
2. **Choose a coordinate system:** Some diagrams on SAT II Physics will provide a coordinate system for you. If they don't, choose one that will simplify your calculations. In this case, let's follow the standard convention of saying that up is the positive y direction and down is the negative y direction.
3. **Draw free-body diagrams:** We know that this pulley system will accelerate when released, so we shouldn't expect the net forces acting on the bodies in the system to be zero. Your free-body diagram should end up looking something like the figure below.



Note that the tension force, T, on each of the blocks is of the same magnitude. In any no stretching rope (the only kind of rope you'll encounter on SAT II Physics), the tension, as well as the velocity and acceleration, is the same at every point. Now, after preparing ourselves to understand the problem, we can begin answering some questions.

1. What is the acceleration of mass M?
2. What is the velocity of mass m after it travels a distance h?
3. What is the work done by the force of tension in lifting mass m a distance h?

1. WHAT IS THE ACCELERATION OF MASS *M*?

Because the acceleration of the rope is of the same magnitude at every point in the rope, the acceleration of the two masses will also be of equal magnitude. If we label the acceleration of mass m as a, then the acceleration of mass M is $-a$. Using Newton's Second Law we find:

$$\text{for mass } M\text{: } T - Mg = -Ma$$
$$\text{for mass } m\text{: } T - mg = ma$$

By subtracting the first equation from the second, we find $(M - m)g = (M + m)a$ or $a = (M - m)g/(M + m)$. Because $M - m > 0$, a is positive and mass m accelerates upward as anticipated. This result gives us a general formula for the acceleration of any pulley system with unequal masses, M and m. Remember, the acceleration is positive for m and negative for M, since m is moving up and M is going down.

$$a = \frac{g(M - m)}{M + m}$$

2. WHAT IS THE VELOCITY OF MASS *M* AFTER IT TRAVELS A DISTANCE *H*?

We could solve this problem by plugging numbers into the kinematics equations, but as you can see, the formula for the acceleration of the pulleys is a bit unwieldy, so the kinematics equations may not be the best approach. Instead, we can tackle this problem in terms of energy. Because the masses in the pulley system are moving up and down, their movement corresponds with a change in gravitational potential energy. Because mechanical energy, E, is conserved, we know that any change in the potential energy, U, of the system will be accompanied by an equal but opposite change in the kinetic energy, KE, of the system.

$$\Delta KE = -\Delta U$$

Remember that since the system begins at rest, $KE_{\text{initial}} = 0$. As the masses move, mass M loses Mgh joules of potential energy, whereas mass m gains mgh joules of potential energy. Applying the law of conservation of mechanical energy, we find:

$$\begin{aligned}\frac{1}{2}Mv^2 + \frac{1}{2}mv^2 &= -(mgh - Mgh)\\ \frac{1}{2}(M+m)v^2 &= (M-m)gh\\ v &= \sqrt{2gh\frac{(M-m)}{M+m}}\end{aligned}$$

Mass m is moving in the positive y direction.

We admit it: the above formula is pretty scary to look at. But since SAT II Physics doesn't allow calculators, you almost certainly will not have to calculate precise numbers for a mass's velocity. It's less important that you have this exact formula memorized, and more important that you understand the principle by which it was derived. You may find a question that involves a derivation of this or some related formula, so it's good to have at least a rough understanding of the relationship between mass, displacement, and velocity in a pulley system.

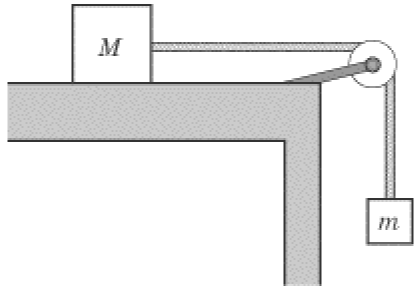
3. WHAT IS THE WORK DONE BY THE FORCE OF TENSION IN LIFTING MASS *M* A DISTANCE *H*?

Since the tension force, $\mathbf{T}$, is in the same direction as the displacement, $\mathbf{h}$, we know that the work done is equal to hT. But what is the magnitude of the tension force? We know that the sum of forces acting on m is $T - mg$ which is equal to ma. Therefore, $T = m(g - a)$. From the solution to question 1, we know that $a = g(M - m)/(M + m)$, so substituting in for a, we get:

$$\begin{aligned}W = hT = m(g-a)h &= m(g - \frac{g(M-m)}{M+m}h)\\ &= mgh(1 - \frac{M-m}{M+m})\end{aligned}$$

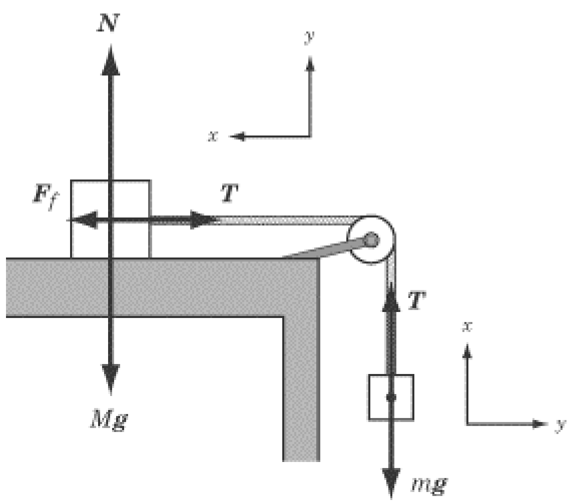
A Pulley on a Table

Now imagine that masses m and M are in the following arrangement:



Let's assume that mass M has already begun to slide along the table, and its movement is opposed by the force of kinetic friction, $F_{fr} = \mu N$, where μ is the coefficient of kinetic friction, and N is the normal force acting between the mass and the table. If the mention of friction and normal forces frightens you, you might want to flip back to Chapter 3 and do a little reviewing. So let's approach this problem with our handy three-step problem-solving method:

1. **Ask yourself how the system will move:** First, we know that mass m is falling and dragging mass M off the table. The force of kinetic friction opposes the motion of mass M. We also know, since both masses are connected by a no stretching rope, that the two masses must have the same velocity and the same acceleration.
2. **Choose a coordinate system:** For the purposes of this problem, it will be easier if we set our coordinate system relative to the rope rather than to the table. If we say that the x-axis runs parallel to the rope, this means the x-axis will be the up-down axis for mass m and the left-right axis for mass M. Further, we can say that gravity pulls in the negative x direction. The y-axis, then, is perpendicular to the rope, and the positive y direction is away from the table.
3. **Draw free-body diagrams:** The above description of the coordinate system may be a bit confusing. That's why a diagram can often be a lifesaver.



Given this information, can you calculate the acceleration of the masses? If you think analytically and don't panic, you can. Since they are attached by a rope, we know that both masses have the same velocity, and hence the same acceleration, a. We also know the net force acting on both masses: the net force acting on mass M is $\mu MG - T$, and the net force acting on mass m is $T - mg$. We can then apply Newton's Second Law to both of the masses, giving us two equations involving a:

$$\text{For mass } M\text{: } \mu MG - T = Ma$$
$$\text{For mass } m\text{: } mg\sin\theta - T = ma$$

Adding the two equations, we find $\mu Mg - mg = (M + m)a$. Solving for a, we get:

$$a = \frac{g(\mu M - m)}{m + M}$$

Since m is moving downward, a must be negative. Therefore, $\mu M < m$.

How Complex Formulas Will Be Tested on SAT II Physics

It is highly unlikely that SAT II Physics will ask a question that involves remembering and then plugging numbers into an equation like this one. Remember: SAT II Physics places far less emphasis on math than your high school physics class. The test writers don't want to test your ability to recall a formula or do some simple math. Rather, they want to determine whether you understand the formulas you've memorized. Here are some examples of the kinds of questions you might be asked regarding the pulley system in the free-body diagram above:

1. **Which of the following five formulas represents the acceleration of the pulley system?** You would then be given five different mathematical formulas, one of which is the correct formula. The test writers would not expect you to have memorized the correct formula, but they would expect you to be able to derive it.
2. **Which of the following is a way of maximizing the system's acceleration?** You would then be given options like "maximize M and m and minimize μ," or "maximize μ and m and minimize M." With such a question, you don't even need to know the correct formula, but you do need to understand how the pulley system works. The downward motion is due to the gravitational force on m and is opposed by the force of friction on M, so we would maximize the downward acceleration by maximizing m and minimizing M and μ
3. **If the system does not move, which of the following must be true?** You would then be given a number of formulas relating M, m, and μ. The idea behind such a question is that the system does not move if the downward force on m is less than or equal to the force of friction on M, so $mg \le \mu Mg$.

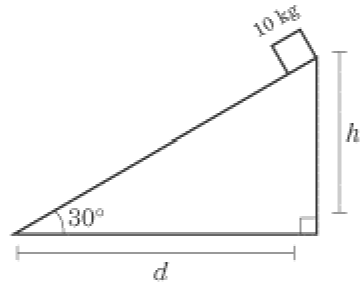
These examples are perhaps less demanding than a question that expects you to derive or recall a complex formula and then plug numbers into it, but they are still difficult questions. In fact, they are about as difficult as mechanics questions on SAT II Physics will get.

Inclined Planes

What we call wedges or slides in everyday language are called **inclined planes** in physics-speak. From our experience on slides during recess in elementary school, sledding down hills in the winter, and skiing, we know that when people are placed on slippery inclines, they slide down the slope. We also know that slides can sometimes be sticky, so that when you are at the top of the incline, you need to give yourself a push to overcome the force of static friction. As you descend a sticky slide, the force of kinetic friction opposes your motion. In this section, we will consider problems involving inclined planes both with and without friction. Since they're simpler, we'll begin with frictionless planes.

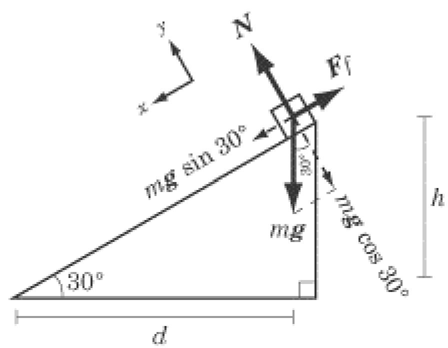
Frictionless Inclined Planes

Suppose you place a 10 kg box on a frictionless 30° inclined plane and release your hold, allowing the box to slide to the ground, a horizontal distance of d meters and a vertical distance of h meters.



Before we continue, let's follow those three important preliminary steps for solving problems in mechanics:

1. **Ask yourself how the system will move:** Because this is a frictionless plane, there is nothing to stop the box from sliding down to the bottom. Experience suggests that the steeper the incline, the faster an object will slide, so we can expect the acceleration and velocity of the box to be affected by the angle of the plane.
2. **Choose a coordinate system:** Because we're interested in how the box slides along the inclined plane, we would do better to orient our coordinate system to the slope of the plane. The x-axis runs parallel to the plane, where downhill is the positive x direction, and the y-axis runs perpendicular to the plane, where up is the positive y direction.
3. **Draw free-body diagrams:** The two forces acting on the box are the force of gravity, acting straight downward, and the normal force, acting perpendicular to the inclined plane, along the y-axis. Because we've oriented our coordinate system to the slope of the plane, we'll have to resolve the vector for the gravitational force, $m\mathbf{g}$, into its x- and y-components. If you recall what we learned about vector decomposition in Chapter 1, you'll know you can break $m\mathbf{g}$ down into a vector of magnitude cos 30° in the negative y direction and a vector of magnitude sin 30° in the positive x direction. The result is a free-body diagram that looks something like this:



Decomposing the $m\mathbf{g}$ vector gives a total of three force vectors at work in this diagram: the y-component of the gravitational force and the normal force, which cancel out; and the x-component of the gravitational force, which pulls the box down the slope. Note that the steeper the slope, the greater the force pulling the box down the slope.

Now let's solve some problems. For the purposes of these problems, take the acceleration due to gravity to be $g = 10$ m/s². Like SAT II Physics, we will give you the values of the relevant trigonometric functions: $\cos 30 = \sin 60 = 0.866$, $\cos 60 = \sin 30 = 0.500$.

1. What is the magnitude of the normal force?
2. What is the acceleration of the box?
3. What is the velocity of the box when it reaches the bottom of the slope?
4. What is the work done on the box by the force of gravity in bringing it to the bottom of the plane?

1. WHAT IS THE MAGNITUDE OF THE NORMAL FORCE?

The box is not moving in the y direction, so the normal force must be equal to the y-component of the gravitational force. Calculating the normal force is then just a matter of plugging a few numbers in for variables in order to find the y-component of the gravitational force:

$$\begin{aligned} N &= mg\cos 30 \\ &= (10\text{ kg})(10\text{ m/s}^2)(0.866) \\ &= 86.6\text{ N} \end{aligned}$$

2. WHAT IS THE ACCELERATION OF THE BOX?

We know that the force pulling the box in the positive x direction has a magnitude of mg sin 30. Using Newton's Second Law, $F = ma$, we just need to solve for a:

$$\begin{aligned} ma &= mg\sin 30 \\ a &= g\sin 30 \\ &= (10\text{ m/s}^2)(0.500) \\ &= 5\text{ m/s}^2 \end{aligned}$$

3. WHAT IS THE VELOCITY OF THE BOX WHEN IT REACHES THE BOTTOM OF THE SLOPE?

Because we're dealing with a frictionless plane, the system is closed and we can invoke the law of conservation of mechanical energy. At the top of the inclined plane, the box will not be moving and so it will have an initial kinetic energy of zero ($KE_{\text{initial}} = 0$). Because it is a height h above the bottom of the plane, it will have a gravitational potential energy of $U = mgh$.

Adding kinetic and potential energy, we find that the mechanical energy of the system is:

$$E = KE + U = 0 + mgh = mgh$$

At the bottom of the slope, all the box's potential energy will have been converted into kinetic energy. In other words, the kinetic energy, $\frac{1}{2}mv^2$, of the box at the bottom of the slope is equal to the potential energy, mgh, of the box at the top of the slope. Solving for v, we get:

$$\begin{aligned} v &= \sqrt{2gh} \\ &= \sqrt{2(10\text{ m/s}^2)h} \\ &= 4.47\sqrt{h} \end{aligned}$$

4. WHAT IS THE WORK DONE ON THE BOX BY THE FORCE OF GRAVITY IN BRINGING IT TO THE BOTTOM OF THE INCLINED PLANE?

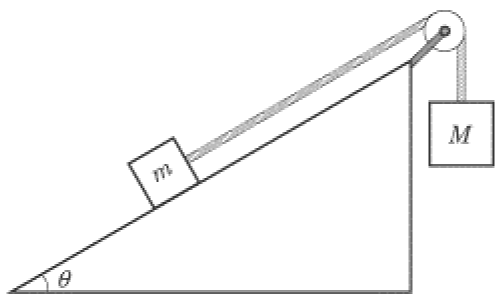
The fastest way to solve this problem is to appeal to the work-energy theorem, which tells us that the work done on an object is equal to its change in kinetic energy. At the top of the slope the box has no kinetic energy, and at the bottom of the slope its kinetic energy is equal to its potential energy at the top of the slope, mgh. So the work done on the box is:

$$\begin{aligned} W &= mgh = (10\text{ kg})(10\text{ m/s}^2)h \\ &= 100h\text{ J} \end{aligned}$$

Note that the work done is independent of how steep the inclined plane is, and is only dependent on the object's change in height when it slides down the plane.

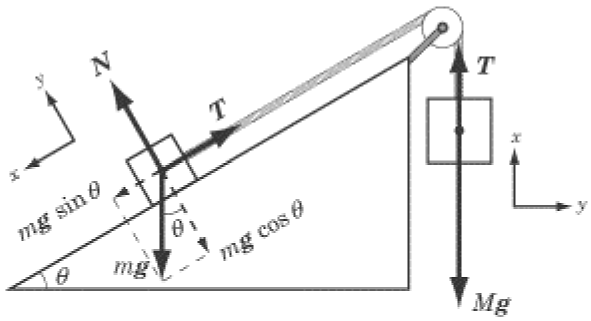
Frictionless Inclined Planes with Pulleys

Let's bring together what we've learned about frictionless inclined planes and pulleys on tables into one exciting problem:



Assume for this problem that $M > m\sin\theta$—that is, mass M will pull mass m up the slope. Now let's ask those three all-important preliminary questions:

1. **Ask yourself how the system will move:** Because the two masses are connected by a rope, we know that they will have the same velocity and acceleration. We also know that the tension in the rope is constant throughout its length. Because $M > m\sin\theta$, we know that when the system is released from rest, mass M will move downward and mass m will slide up the inclined plane.
2. **Choose a coordinate system:** Do the same thing here that we did with the previous pulley-on-a-table problem. Make the x-axis parallel to the rope, with the positive x direction being up for mass M and downhill for mass m, and the negative x direction being down for mass M and uphill for mass m. Make the y-axis perpendicular to the rope, with the positive y-axis being away from the inclined plane, and the negative y-axis being toward the inclined plane.
3. **Draw free-body diagrams:** We've seen how to draw free-body diagrams for masses suspended from pulleys, and we've seen how to draw free-body diagrams for masses on inclined planes. All we need to do now is synthesize what we already know:



Now let's tackle a couple of questions:

1. What is the acceleration of the masses?
2. What is the velocity of mass m after mass M has fallen a distance h?

1. WHAT IS THE ACCELERATION OF THE MASSES?

First, let's determine the net force acting on each of the masses. Applying Newton's Second Law we get:

$$\text{for mass } M\text{: } \mu Mg - T = Ma$$
$$\text{for mass } m\text{: } T - mg = ma$$

Adding these two equations together, we find that $mg\sin\theta - Mg = (m+M)a$. Solving for a, we get:

$$a = \frac{g(m\sin\theta - M)}{(m+M)}$$

Because $M > m\sin\theta$, the acceleration is negative, which, as we defined it, is down for mass M and uphill for mass m.

2. WHAT IS THE VELOCITY OF MASS *M* AFTER MASS *M* HAS FALLEN A DISTANCE *H*?

Once again, the inclined plane is frictionless, so we are dealing with a closed system and we can apply the law of conservation of mechanical energy. Since the masses are initially at rest, $KE_{\text{initial}} = 0$. Since mass M falls a distance h, its potential energy changes by $-Mgh$. If mass M falls a distance h, then mass m must slide the same distance up the slope of the inclined plane, or a vertical distance of $h\sin\theta$. Therefore, mass m's potential energy increases by $mgh\sin\theta$. Because the sum of potential energy and kinetic energy cannot change, we know that the kinetic energy of the two masses increases precisely to the extent that their potential energy decreases. We have all we need to scribble out some equations and solve for v:

$$\frac{1}{2}Mv^2 + \frac{1}{2}mv^2 - Mgh + mgh\sin\theta = 0$$
$$\frac{1}{2}(m+M)v^2 = gh(M - m\sin\theta)$$
$$v = \sqrt{\frac{2gh(M - m\sin\theta)}{m+M}}$$

Finally, note that the velocity of mass m is in the uphill direction.

As with the complex equations we encountered with pulley systems above, you needn't trouble yourself with memorizing a formula like this. If you understand the principles at work in this problem and would feel somewhat comfortable deriving this formula, you know more than SAT II Physics will likely ask of you.

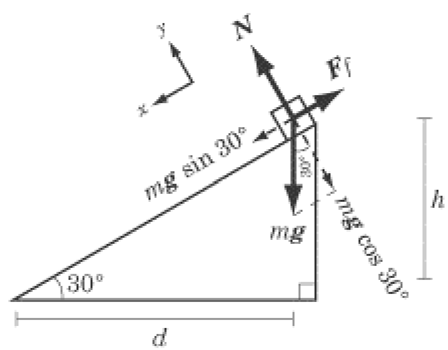
Inclined Planes With Friction

There are two significant differences between frictionless inclined plane problems and inclined plane problems where friction is a factor:

1. **There's an extra force to deal with.** The force of friction will oppose the downhill component of the gravitational force.
2. **We can no longer rely on the law of conservation of mechanical energy.** Because energy is being lost through the friction between the mass and the inclined plane, we are no longer dealing with a closed system. Mechanical energy is not conserved.

Consider the 10 kg box we encountered in our example of a frictionless inclined plane. This time, though, the inclined plane has a coefficient of kinetic friction of $\mu = 0.5$. How will this additional factor affect us? Let's follow three familiar steps:

1. **Ask yourself how the system will move:** If the force of gravity is strong enough to overcome the force of friction, the box will accelerate down the plane. However, because there is a force acting against the box's descent, we should expect it to slide with a lesser velocity than it did in the example of the frictionless plane.
2. **Choose a coordinate system:** There's no reason not to hold onto the co-ordinate system we used before: the positive *x* direction is down the slope, and the positive *y* direction is upward, perpendicular to the slope.
3. **Draw free-body diagrams:** The free-body diagram will be identical to the one we drew in the example of the frictionless plane, except we will have a vector for the force of friction in the negative *x* direction.



Now let's ask some questions about the motion of the box.

1. What is the force of kinetic friction acting on the box?
2. What is the acceleration of the box?
3. What is the work done on the box by the force of kinetic friction?

WHAT IS THE FORCE OF KINETIC FRICTION ACTING ON THE BOX?

The normal force acting on the box is 86.6 N, exactly the same as for the frictionless inclined plane. The force of kinetic friction is defined as $F_f = \mu N$, so plugging in the appropriate values for μ and N:

$$\begin{aligned} F_f &= \mu N = 0.500 \cdot 86.6 \text{ N} \\ &= 43.3 \text{ N} \end{aligned}$$

Remember, though, that the force of friction is exerted in the negative x direction, so the correct answer is –43.3 N.

WHAT IS THE ACCELERATION OF THE BOX?

The net force acting on the box is the difference between the downhill gravitational force and the force of friction: $F = mg \sin 30 - F_f$. Using Newton's Second Law, we can determine the net force acting on the box, and then solve for a:

$$\begin{aligned} ma &= mg \sin 30 - F_f \\ (10 \text{ kg})a &= (10 \text{ kg})(10 \text{ m/s}^2)(0.500) - 43.3 \text{ N} \\ a &= \frac{50 \text{ N} - 43.3 \text{ N}}{10 \text{ kg}} \\ &= 0.67 \text{ m/s}^2 \end{aligned}$$

Because $mg \sin 30 > F_f$, the direction of the acceleration is in the downhill direction.

WHAT IS THE WORK DONE ON THE BOX BY THE FORCE OF KINETIC FRICTION?

Since $W = \mathbf{F} \cdot \mathbf{d}$, the work done by the force of friction is the product of the force of friction and the displacement of the box in the direction that the force is exerted. Because the force of friction is exerted in the negative x direction, we need to find the displacement of the box in the x direction. We know that it has traveled a horizontal distance of d and a vertical distance of h. The Pythagorean Theorem then tells us that the displacement of the box is $\sqrt{d^2 + h^2}$. Recalling that the force of friction is –43.3 N, we know that the work done by the force of friction is

$$W = -43.3\sqrt{d^2 + h^2} \text{ J}$$

Note that the amount of work done is negative, because the force of friction acts in the opposite direction of the displacement of the box.

Springs

Questions about springs on SAT II Physics are usually simple matters of a mass on a spring oscillating back and forth. However, spring motion is the most interesting of the four topics we will cover here because of its generality. The **harmonic motion** that springs exhibit applies equally to objects moving in a circular path and to the various wave phenomena that we'll study later in this book. So before we dig in to the nitty-gritty of your typical SAT II Physics spring questions, let's look at some general features of harmonic motion.

Oscillation and Harmonic Motion

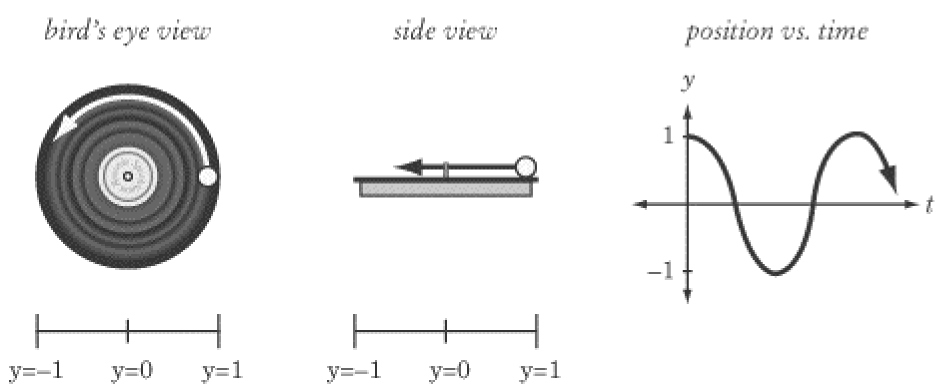
Consider the following physical phenomena:

- When you drop a rock into a still pond, the rock makes a big splash, which causes ripples to spread out to the edges of the pond.
- When you pluck a guitar string, the string vibrates back and forth.
- When you rock a small boat, it wobbles to and fro in the water before coming to rest again.
- When you stretch out a spring and release it, the spring goes back and forth between being compressed and being stretched out.

There are just a few examples of the widespread phenomenon of **oscillation**. Oscillation is the natural world's way of returning a system to its **equilibrium position**, the stable position of the system where the net force acting on it is zero. If you throw a system off-balance, it doesn't simply return to the way it was; it oscillates back and forth about the equilibrium position.

A system oscillates as a way of giving off energy. A system that is thrown off-kilter has more energy than a system in its equilibrium position. To take the simple example of a spring, a stretched-out spring will start to move as soon as you let go of it: that motion is evidence of kinetic energy that the spring lacks in its equilibrium position. Because of the law of conservation of energy, a stretched-out spring cannot simply return to its equilibrium position; it must release some energy in order to do so. Usually, this energy is released as thermal energy caused by friction, but there are plenty of interesting exceptions. For instance, a plucked guitar string releases sound energy: the music we hear is the result of the string returning to its equilibrium position.

The movement of an oscillating body is called harmonic motion. If you were to graph the position, velocity, or acceleration of an oscillating body against time, the result would be a sinusoidal wave; that is, some variation of a $y = a \sin bx$ or a $y = a \cos bx$ graph. This generalized form of harmonic motion applies not only to springs and guitar strings, but to anything that moves in a **cycle**. Imagine placing a pebble on the edge of a turntable, and watching the turntable rotate while looking at it from the side. You will see the pebble moving back and forth in one dimension. The pebble will appear to oscillate just like a spring: it will appear to move fastest at the middle of its trajectory and slow to a halt and reverse direction as it reaches the edge of its trajectory.

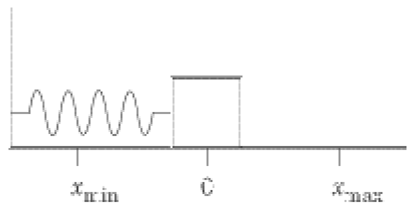


This example serves two purposes. First, it shows you that the oscillation of springs is just one of a wide range of phenomena exhibiting harmonic motion. Anything that moves in a cyclic pattern exhibits harmonic motion. This includes the light and sound waves without which we would have a lot of trouble moving about in the world. Second, we bring it up because SAT II Physics has been known to test students on the nature of the horizontal or vertical component of the motion of an object in circular motion. As you can see, circular motion viewed in one dimension is harmonic motion.

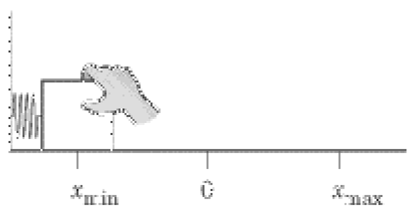
Though harmonic motion is one of the most widespread and important of physical phenomena, your understanding of it will not be taxed to any great extent on SAT II Physics. In fact, beyond the motion of springs and pendulums, everything you will need to know will be covered in this book in the chapter on Waves. The above discussion is mostly meant to fit your understanding of the oscillation of springs into a wider context.

The Oscillation of a Spring

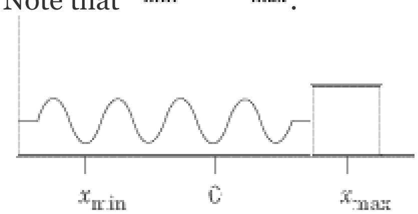
Now let's focus on the harmonic motion exhibited by a spring. To start with, we'll imagine a mass, m, placed on a frictionless surface, and attached to a wall by a spring. In its equilibrium position, where no forces act upon it, the mass is at rest. Let's label this equilibrium position $x = 0$. Intuitively, you know that if you compress or stretch out the spring it will begin to oscillate.



Suppose you push the mass toward the wall, compressing the spring, until the mass is in position $x = x_{min}$.



When you release the mass, the spring will exert a force, pushing the mass back until it reaches position $x = x_{max}$, which is called the **amplitude** of the spring's motion, or the maximum displacement of the oscillator. Note that $x_{min} = -x_{max}$.



By that point, the spring will be stretched out, and will be exerting a force to pull the mass back in toward the wall. Because we are dealing with an idealized frictionless surface, the mass will not be slowed by the force of friction, and will oscillate back and forth repeatedly between x_{min} and x_{max}.

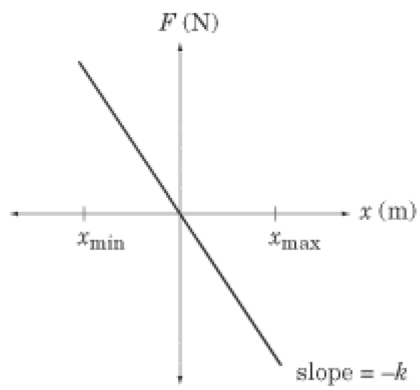
Hooke's Law

This is all well and good, but we can't get very far in sorting out the amplitude, the velocity, the energy, or anything else about the mass's motion if we don't understand the manner in which the spring exerts a force on the mass attached to it. The force, $\boldsymbol{F}$, that the spring exerts on the mass is defined by **Hooke's Law**:

$$\boldsymbol{F} = -k\boldsymbol{x}$$

where $\boldsymbol{x}$ is the spring's displacement from its equilibrium position and k is a constant of proportionality called the **spring constant**. The spring constant is a measure of "springiness": a greater value for k signifies a "tighter" spring, one that is more resistant to being stretched.

Hooke's Law tells us that the further the spring is displaced from its equilibrium position ($\boldsymbol{x}$) the greater the force the spring will exert in the direction of its equilibrium position ($\boldsymbol{F}$). We call $\boldsymbol{F}$ a **restoring force**: it is always directed toward equilibrium. Because F and x are directly proportional, a graph of F vs. x is a line with slope $-k$.



Simple Harmonic Oscillation

A mass oscillating on a spring is one example of a **simple harmonic oscillator**. Specifically, a simple harmonic oscillator is any object that moves about a stable equilibrium point and experiences a restoring force proportional to the oscillator's displacement.

For an oscillating spring, the restoring force, and consequently the acceleration, are greatest and positive at x_{min}. These quantities decrease as x approaches the equilibrium position and are zero at $x = 0$. The restoring force and acceleration—which are now negative—increase in magnitude as x approaches x_{max} and are maximally negative at x_{max}.

Important Properties of a Mass on a Spring

There are a number of important properties related to the motion of a mass on a spring, all of which are fair game for SAT II Physics. Remember, though: the test makers have no interest in testing your ability to recall complex formulas and perform difficult mathematical operations. You may be called upon to know the simpler of these formulas, but not the complex ones. As we mentioned at the end of the section on pulleys, it's less important that you memorize the formulas and more important that you understand what they mean. If you understand the principle, there probably won't be any questions that will stump you.

Period of Oscillation

The period of oscillation, T, of a spring is the amount of time it takes for a spring to complete a round-trip or cycle. Mathematically, the period of oscillation of a simple harmonic oscillator described by Hooke's Law is:

$$T = 2\pi\sqrt{\frac{m}{k}}$$

This equation tells us that as the mass of the block, m, increases and the spring constant, k, decreases, the period increases. In other words, a heavy mass attached to an easily stretched spring will oscillate back and forth very slowly, while a light mass attached to a resistant spring will oscillate back and forth very quickly.

Frequency

The frequency of the spring's motion tells us how quickly the object is oscillating, or how many cycles it completes in a given timeframe. Frequency is inversely proportional to period:

$$f = \frac{1}{T}$$

Frequency is given in units of cycles per second, or hertz (Hz).

Potential Energy

The potential energy of a spring (U_s) is sometimes called elastic energy, because it results from the spring being stretched or compressed. Mathematically, U_s is defined by:

$$U_s = \frac{1}{2}kx^2$$

The potential energy of a spring is greatest when the coil is maximally compressed or stretched, and is zero at the equilibrium position.

Kinetic Energy

SAT II Physics will not test you on the motion of springs involving friction, so for the purposes of the test, the mechanical energy of a spring is a conserved quantity. As we recall, mechanical energy is the sum of the kinetic energy and potential energy.

At the points of maximum compression and extension, the velocity, and hence the kinetic energy, is zero and the mechanical energy is equal to the potential energy, $U_s = {}^1/_2\, kx^2$.

At the equilibrium position, the potential energy is zero, and the velocity and kinetic energy are maximized. The kinetic energy at the equilibrium position is equal to the mechanical energy:

$$KE_{\text{max}} = \frac{1}{2}mv^2 = \frac{1}{2}kx^2_{\text{max}}$$

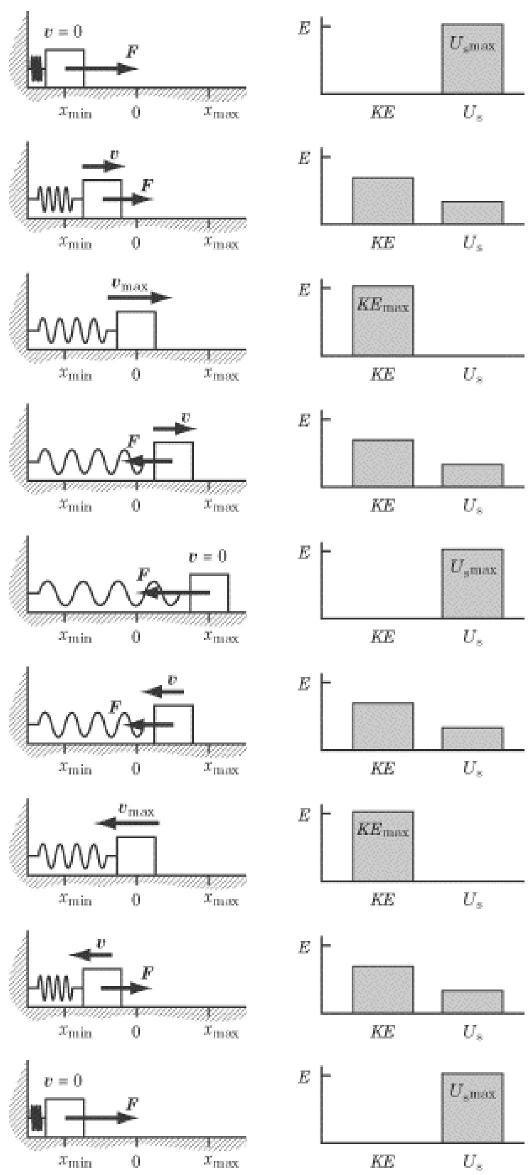
From this equation, we can derive the maximum velocity:

$$v_{max} = x_{max}\sqrt{\frac{k}{m}}$$

You won't need to know this equation, but it might be valuable to note that the velocity increases with a large displacement, a resistant spring, and a small mass.

Summary

It is highly unlikely that the formulas discussed above will appear on SAT II Physics. More likely, you will be asked conceptual questions such as: at what point in a spring's oscillation is the kinetic or potential energy maximized or minimized, for instance. The figure below summarizes and clarifies some qualitative aspects of simple harmonic oscillation. Your qualitative understanding of the relationship between force, velocity, and kinetic and potential energy in a spring system is far more likely to be tested than your knowledge of the formulas discussed above.



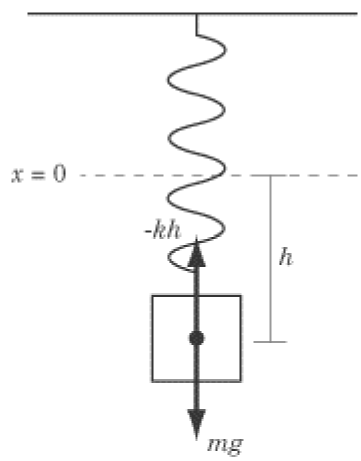
In this figure, v represents velocity, F represents force, KE represents kinetic energy, and U_s represents potential energy.

Vertical Oscillation of Springs

Now let's consider a mass attached to a spring that is suspended from the ceiling. Questions of this sort have a nasty habit of coming up on SAT II Physics. The oscillation of the spring when compressed or extended won't be any different, but we now have to take gravity into account.

Equilibrium Position

Because the mass will exert a gravitational force to stretch the spring downward a bit, the equilibrium position will no longer be at $x = 0$, but at $x = -h$, where h is the vertical displacement of the spring due to the gravitational pull exerted on the mass. The equilibrium position is the point where the net force acting on the mass is zero; in other words, the point where the upward restoring force of the spring is equal to the downward gravitational force of the mass.



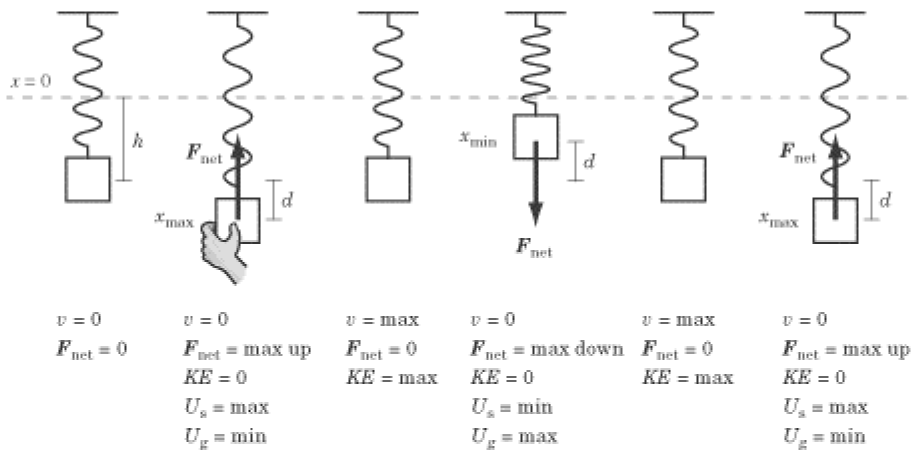
Combining the restoring force, $F = -kh$, and the gravitational force, $F = mg$, we can solve for h:

$$-kh = mg$$
$$-h = \frac{mg}{k}$$

Since m is in the numerator and k in the denominator of the fraction, the mass displaces itself more if it has a large weight and is suspended from a lax spring, as intuition suggests.

A Vertical Spring in Motion

If the spring is then stretched a distance d, where $d < h$, it will oscillate between $x_{\max} = -h - d$ and $x_{\min} = -h + d$.



Throughout the motion of the mass, the force of gravity is constant and downward. The restoring force of the spring is always upward, because even at x_{min} the mass is below the spring's initial equilibrium position of $x = 0$. Note that if d were greater than h, x_{min} would be above $x = 0$, and the restoring force would act in the downward direction until the mass descended once more below $x = 0$.

According to Hooke's Law, the restoring force decreases in magnitude as the spring is compressed. Consequently, the net force downward is greatest at $x = x_{min}$ and the net force upward is greatest at $x = x_{max}$.

Energy

The mechanical energy of the vertically oscillating spring is:

$$E = KE + U_g + U_s$$

where U_g is gravitational potential energy and U_s is the spring's (elastic) potential energy. Note that the velocity of the block is zero at $x = x_{min}$ and $x = x_{max}$, and maximized at the equilibrium position, $x = -h$. Consequently, the kinetic energy of the spring is zero for $x = x_{max}$ and $x = x_{min}$ and is greatest at $x = -h$. The gravitational potential energy of the system increases with the height of the mass. The elastic potential energy of the spring is greatest when the spring is maximally extended at x_{max} and decreases with the extension of the spring.

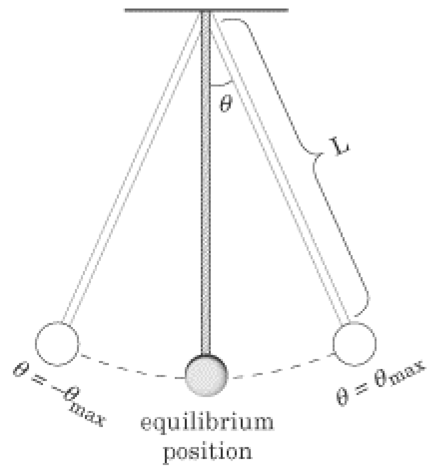
How This Knowledge Will Be Tested

Most of the questions on SAT II Physics that deal with spring motion will ask qualitatively about the energy or velocity of a vertically oscillating spring. For instance, you may be shown a diagram capturing one moment in a spring's trajectory and asked about the relative magnitudes of the gravitational and elastic potential energies and kinetic energy. Or you may be asked at what point in a spring's trajectory the velocity is maximized. The answer, of course, is that it is maximized at the equilibrium position. It is far less likely that you will be asked a question that involves any sort of calculation.

Pendulums

A **pendulum** is defined as a mass, or bob, connected to a rod or rope, that experiences simple harmonic motion as it swings back and forth without friction. The equilibrium position of the pendulum is the position when the mass is hanging directly downward.

Consider a pendulum bob connected to a mass less rope or rod that is held at an angle θ_{max} from the horizontal. If you release the mass, then the system will swing to position $-\theta_{max}$ and back again.



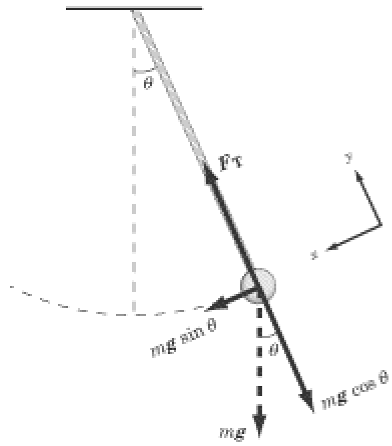
The oscillation of a pendulum is much like that of a mass on a spring. However, there are significant differences, and many a student has been tripped up by trying to apply the principles of a spring's motion to pendulum motion.

Properties of Pendulum Motion

As with springs, there are a number of properties of pendulum motion that you might be tested on, from frequency and period to kinetic and potential energy. Let's apply our three-step method of approaching special problems in mechanics and then look at the formulas for some of those properties:

1. **Ask yourself how the system will move:** It doesn't take a rocket scientist to surmise that when you release the pendulum bob it will accelerate toward the equilibrium position. As it passes through the equilibrium position, it will slow down until it reaches position $-\theta$, and then accelerate back. At any given moment, the velocity of the pendulum bob will be perpendicular to the rope. The pendulum's trajectory describes an arc of a circle, where the rope is a radius of the circle and the bob's velocity is a line tangent to the circle.
2. **Choose a coordinate system:** We want to calculate the forces acting on the pendulum at any given point in its trajectory. It will be most convenient to choose a y-axis that runs parallel to the rope. The x-axis then runs parallel to the instantaneous velocity of the bob so that, at any given moment, the bob is moving along the x-axis.
3. **Draw free-body diagrams:** Two forces act on the bob: the force of gravity, $\mathbf{F} = m\mathbf{g}$, pulling the bob straight downward and the tension of the rope, $\mathbf{F}_T$, pulling the bob upward along the y-axis. The gravitational force can be broken down into an x-component, $mg \sin \theta$, and a y-component, $mg \cos \theta$. The y component balances out the force of tension—the

pendulum bob doesn't accelerate along the y-axis—so the tension in the rope must also be $mg \cos \theta$. Therefore, the tension force is maximum for the equilibrium position and decreases with θ. The restoring force is $mg \sin \theta$, so, as we might expect, the restoring force is greatest at the endpoints of the oscillation, $\theta = \pm\theta_{\max}$ and is zero when the pendulum passes through its equilibrium position.



You'll notice that the restoring force for the pendulum, $mg \sin \theta$, is not directly proportional to the displacement of the pendulum bob, θ, which makes calculating the various properties of the pendulum very difficult. Fortunately, pendulums usually only oscillate at small angles, where $\sin \theta \approx \theta$. In such cases, we can derive more straightforward formulas, which are admittedly only approximations. However, they're good enough for the purposes of SAT II Physics.

Period

The period of oscillation of the pendulum, T, is defined in terms of the acceleration due to gravity, g, and the length of the pendulum, L:

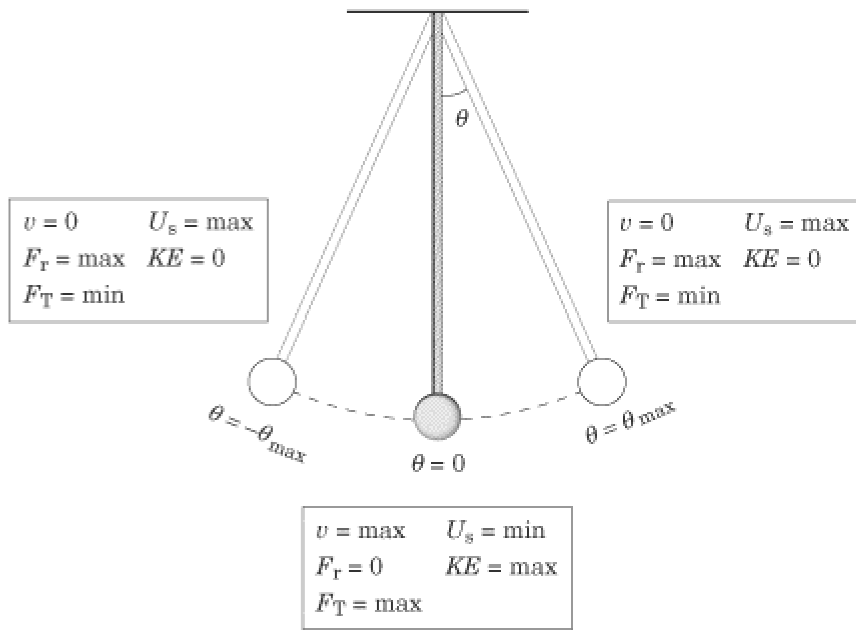
$$T = 2\pi\sqrt{\frac{L}{g}}$$

This is a pretty scary-looking equation, but there's really only one thing you need to gather from it: the longer the pendulum rope, the longer it will take for the pendulum to oscillate back and forth. You should also note that the mass of the pendulum bob and the angle of displacement play no role in determining the period of oscillation.

Energy

The mechanical energy of the pendulum is a conserved quantity. The potential energy of the pendulum, mgh, increases with the height of the bob; therefore the potential energy is minimized at the equilibrium point and is maximized at $\theta = \pm\theta_{\max}$. Conversely, the kinetic energy and velocity of the pendulum are maximized at the equilibrium point and minimized when $\theta = \pm\theta_{\max}$.

The figure below summarizes this information in a qualitative manner, which is the manner in which you are most likely to find it on SAT II Physics. In this figure, v signifies velocity, F_r signifies the restoring force, F_T signifies the tension in the pendulum string, U signifies potential energy, and KE signifies kinetic energy.

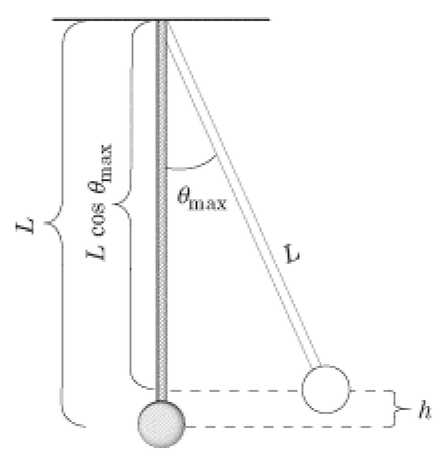


Velocity

Calculating the velocity of the pendulum bob at the equilibrium position requires that we arrange our coordinate system so that the height of the bob at the equilibrium position is zero. Then the total mechanical energy is equal to the kinetic energy at the equilibrium point where $U = 0$. The total mechanical energy is also equal to the total potential energy at $\pm\theta_{max}$ where $KE = 0$. Putting these equalities together, we get

$$E = \frac{1}{2}mv^2 = mgh$$

But what is h?



From the figure, we see that $h = L - L\cos(-\theta_{max})$. If we plug that value into the equation above, we can solve for *v*:

$$\frac{1}{2}mv^2 = mgL(1 - \cos\theta_{max})$$
$$v = \sqrt{2gL(1 - \cos\theta_{max})}$$

Don't let a big equation frighten you. Just register what it conveys: the longer the string and the greater the angle, the faster the pendulum bob will move.

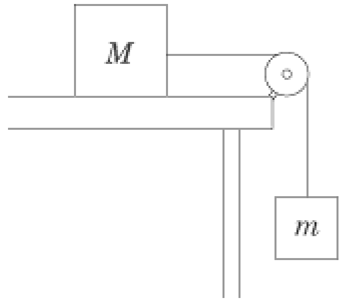
How This Knowledge Will Be Tested

Again, don't worry too much about memorizing equations: most of the questions on pendulum motion will be qualitative. There may be a question asking you at what point the tension in the rope is greatest (at the equilibrium position) or where the bob's potential energy is maximized (at $\theta = \pm\theta_{max}$). It's highly unlikely that you'll be asked to give a specific number.

Key Formulas

Hooke's Law	$\boldsymbol{F} = -k\boldsymbol{x}$
Period of Oscillation of a Spring	$T = 2\pi\sqrt{\frac{m}{k}}$
Frequency	$f = \frac{1}{T}$
Potential Energy of a Spring	$U_s = \frac{1}{2}kx^2$
Velocity of a Spring at the Equilibrium Position	$v_{max} = x\sqrt{\frac{k}{m}}$
Period of Oscillation of a Pendulum	$T = 2\pi\sqrt{\frac{L}{g}}$
Velocity of a Pendulum Bob at the Equilibrium Position	$\frac{1}{2}mv^2 = mgL(1 - \cos\theta_{max})$ $v = \sqrt{2gL(1 - \cos\theta_{max})}$

Practice Questions



1. Two masses, *m* and *M*, are connected to a pulley system attached to a table, as in the diagram above. What is the minimum value for the coefficient of static friction between mass *M* and the table if the pulley system does not move?

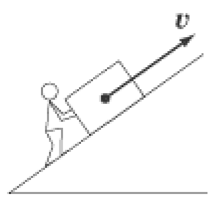
   (A) m/M

   (B) M/m

   (C) $g\,(m/M)$

   (D) $g\,(M/m)$

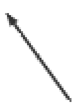
   (E) $g(M - m)$

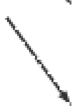


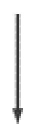
2. A mover pushes a box up an inclined plane, as shown in the figure above. Which of the following shows the direction of the normal force exerted by the plane on the box?

   (A) 

   (B) 

   (C) 

   (D) 

   (E) 

3. Consider a block sliding down a frictionless inclined plane with acceleration *a*. If we double the mass of the block, what is its acceleration?

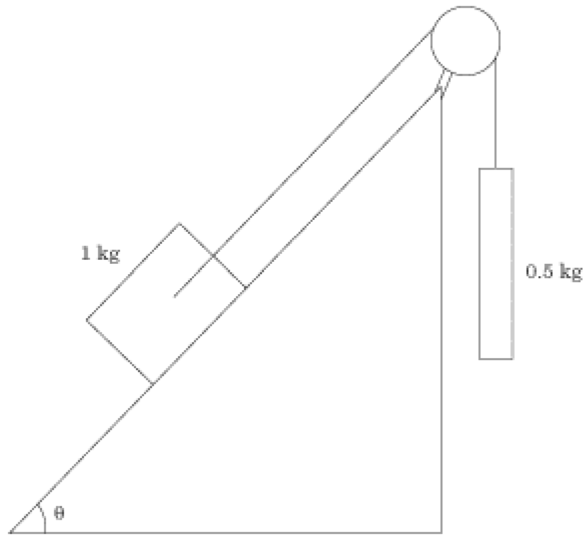
   (A) $a/4$

   (B) $a/2$

   (C) a

   (D) $2a$

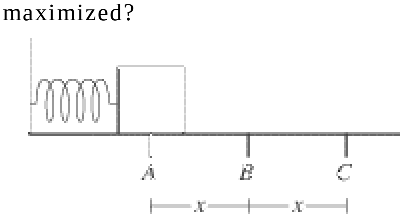
   (E) $4a$



4. A 1 kg mass on a frictionless inclined plane is connected by a pulley to a hanging 0.5 kg mass, as in the diagram above. At what angle will the system be in equilibrium? cos 30Â° = sin 60Â° = $\sqrt{3}$, cos 60Â° = sin 30Â° = 1/2, cos 45Â° = sin 45Â° = 1/ $\sqrt{2}$.
   (A) 0Â°
   (B) –30Â°
   (C) 30Â°
   (D) 45Â°
   (E) 60Â°

5. An object of mass *m* rests on a plane inclined at an angle of θ. What is the maximum value for the coefficient of static friction at which the object will slide down the incline?
   (A) $\sin\theta - \cos\theta$
   (B) $\cos\theta - \sin\theta$
   (C) $mg\sin\theta$
   (D) $\sin\theta / \cos\theta$
   (E) $\sin\theta + \cos\theta$

6. A mass on a frictionless surface is attached to a spring. The spring is compressed from its equilibrium position, *B*, to point *A*, a distance *x* from *B*. Point *C* is also a distance *x* from *B*, but in the opposite direction. When the mass is released and allowed to oscillated freely, at what point or points is its velocity maximized?



   (A) *A*
   (B) *B*
   (C) *C*
   (D) Both *A* and *C*

(E) Both *A* and *B*

7. An object of mass 3 kg is attached to a spring of spring constant 50 N/m. How far is the equilibrium position of this spring system from the point where the spring exerts no force on the object?

   (A) 0.15 m

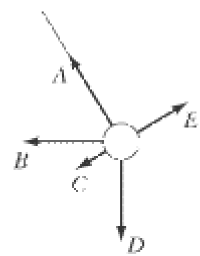
   (B) 0.3 m

   (C) 0.5 m

   (D) 0.6 m

   (E) 1.5 m

Questions 8–10 refer to a pendulum in its upward swing. That is, the velocity vector for the pendulum is pointing in the direction of *E*.



8. What is the direction of the force of gravity on the pendulum bob?

   (A) *A*

   (B) *B*

   (C) *C*

   (D) *D*

   (E) *E*

9. What is the direction of the net force acting on the pendulum?

   (A) *A*

   (B) *B*

   (C) *C*

   (D) *D*

   (E) *E*

10. If the pendulum string is suddenly cut, what is the direction of the velocity vector of the pendulum bob the moment it is released?

    (A) *A*

    (B) *B*

    (C) *C*

    (D) *D*

    (E) *E*

Linear Momentum

THE CONCEPT OF **linear momentum** IS closely tied to the concept of force—in fact, Newton first defined his Second Law not in terms of mass and acceleration, but in terms of momentum. Like energy, linear momentum is a conserved quantity in closed systems, making it a very handy tool for solving problems in mechanics. On the whole, it is useful to analyze systems in terms of energy when there is an exchange of potential energy and kinetic energy. Linear momentum, however, is useful in those cases where there is no clear measure for potential energy. In particular, we will use the law of **conservation of momentum** to determine the outcome of collisions between two bodies.

What Is Linear Momentum?

Linear momentum is a vector quantity defined as the product of an object's mass, m, and its velocity, $\mathbf{v}$. Linear momentum is denoted by the letter $\mathbf{p}$ and is called "momentum" for short:

$$\mathbf{p} = m\mathbf{v}$$

Note that a body's momentum is always in the same direction as its velocity vector. The units of momentum are kg · m/s.

Fortunately, the way that we use the word *momentum* in everyday life is consistent with the definition of momentum in physics. For example, we say that a BMW driving 20 miles per hour has less momentum than the same car speeding on the highway at 80 miles per hour. Additionally, we know that if a large truck and a BMW travel at the same speed on a highway, the truck has a greater momentum than the BMW, because the truck has greater mass. Our everyday usage reflects the definition given above, that momentum is proportional to mass and velocity.

Linear Momentum and Newton's Second Law

In Chapter 3, we introduced Newton's Second Law as $\mathbf{F} = m\mathbf{a}$. However, since acceleration can be expressed as $\Delta\mathbf{v}/\Delta t$, we could equally well express Newton's Second Law as $\mathbf{F} = m\Delta\mathbf{v}/\Delta t$. Substituting $\mathbf{p}$ for $m\mathbf{v}$, we find an expression of Newton's Second Law in terms of momentum:

$$\mathbf{F} = \frac{\Delta\mathbf{p}}{\Delta t}$$

In fact, this is the form in which Newton first expressed his Second Law. It is more flexible than $\mathbf{F} = m\mathbf{a}$ because it can be used to analyze systems where not just the velocity, but also the mass of a body changes, as in the case of a rocket burning fuel.

Impulse

The above version of Newton's Second Law can be rearranged to define the **impulse**, $\mathbf{J}$, delivered by a constant force, $\mathbf{F}$. Impulse is a vector quantity defined as the product of the force acting on a body and the time interval during which the force is exerted. If the force changes during the time interval, $\mathbf{F}$ is the average net force over that time interval. The impulse caused by a force during a specific time interval is equal to the body's change of momentum during that time interval: impulse, effectively, is a measure of change in momentum.

$$J = F\Delta t = \Delta p$$

The unit of impulse is the same as the unit of momentum, kg · m/s.

EXAMPLE

> A soccer player kicks a 0.1 kg ball that is initially at rest so that it moves with a velocity of 20 m/s. What is the impulse the player imparts to the ball? If the player's foot was in contact with the ball for 0.01 s, what was the force exerted by the player's foot on the ball?

What is the impulse the player imparts to the ball?

Since impulse is simply the change in momentum, we need to calculate the difference between the ball's initial momentum and its final momentum. Since the ball begins at rest, its initial velocity, and hence its initial momentum, is zero. Its final momentum is:

$$\begin{aligned} p &= mv = (0.1 \text{ kg})(20 \text{ m/s}) \\ &= 2 \text{ kg} \cdot \text{m/s} \end{aligned}$$

Because the initial momentum is zero, the ball's change in momentum, and hence its impulse, is 2 kg · m/s.

What was the force exerted by the player's foot on the ball?

Impulse is the product of the force exerted and the time interval over which it was exerted. It follows, then, that $F = J/\Delta t$. Since we have already calculated the impulse and have been given the time interval, this is an easy calculation:

$$\begin{aligned} F &= \frac{J}{\Delta t} = \frac{2 \text{ kg} \cdot \text{m/s}}{0.01 \text{ s}} \\ &= 200 \text{ N} \end{aligned}$$

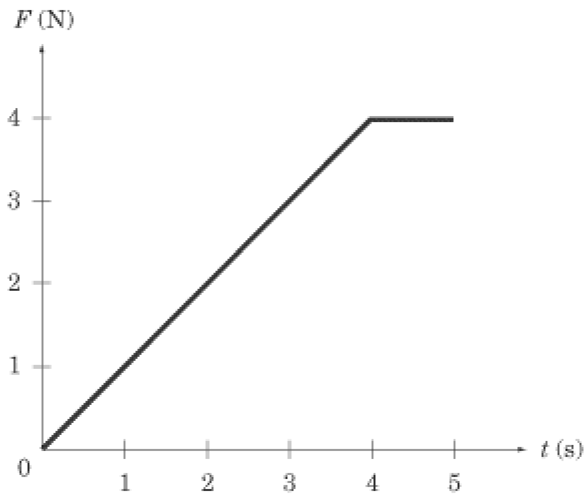
Impulse and Graphs

SAT II Physics may also present you with a force vs. time graph, and ask you to calculate the impulse. There is a single, simple rule to bear in mind for calculating the impulse in force vs. time graphs:

The impulse caused by a force during a specific time interval is equal to the area underneath the force vs. time graph during the same interval.

If you recall, whenever you are asked to calculate the quantity that comes from multiplying the units measured by the *y*-axis with the units measured by the *x*-axis, you do so by calculating the area under the graph for the relevant interval.

EXAMPLE



What is the impulse delivered by the force graphed in the figure above between $t = 0$ and $t = 5$?

The impulse over this time period equals the area of a triangle of height 4 and base 4plus the area of a rectangle of height 4 and width 1. A quick calculation shows us that the impulse is:

$$\boldsymbol{J} = \frac{1}{2} \cdot 4 \cdot 4 + 4 \cdot 1 = 12\ \mathrm{kg} \cdot \mathrm{m/s}$$

Conservation of Momentum

If we combine Newton's Third Law with what we know about impulse, we can derive the important and extremely useful law of conservation of momentum.

Newton's Third Law tells us that, to every action, there is an equal and opposite reaction. If object A exerts a force $\boldsymbol{F}$ on object B, then object B exerts a force $-\boldsymbol{F}$ on object A. The net force exerted between objects A and B is zero.

The impulse equation, $\boldsymbol{J} = \boldsymbol{F}\Delta t = \Delta \boldsymbol{p}$, tells us that if the net force acting on a system is zero, then the impulse, and hence the change in momentum, is zero. Because the net force between the objects A and B that we discussed above is zero, the momentum of the system consisting of objects A and B does not change.

Suppose object A is a cue ball and object B is an eight ball on a pool table. If the cue ball strikes the eight ball, the cue ball exerts a force on the eight ball that sends it rolling toward the pocket. At the same time, the eight ball exerts an equal and opposite force on the cue ball that brings it to a stop. Note that both the cue ball and the eight ball each experience a change in momentum. However, the sum of the momentum of the cue ball and the momentum of the eight ball remains constant throughout. While the initial momentum of the cue ball, $\boldsymbol{p}_A$, is not the same as its final momentum, $\boldsymbol{p}_A'$, and the initial momentum of the eight ball, $\boldsymbol{p}_B$, is not the same as its final momentum, $\boldsymbol{p}_B'$, the initial momentum of the two balls combined is equal to the final momentum of the two balls combined:

$$\boldsymbol{p}_A + \boldsymbol{p}_B = \boldsymbol{p}_A' + \boldsymbol{p}_B'$$

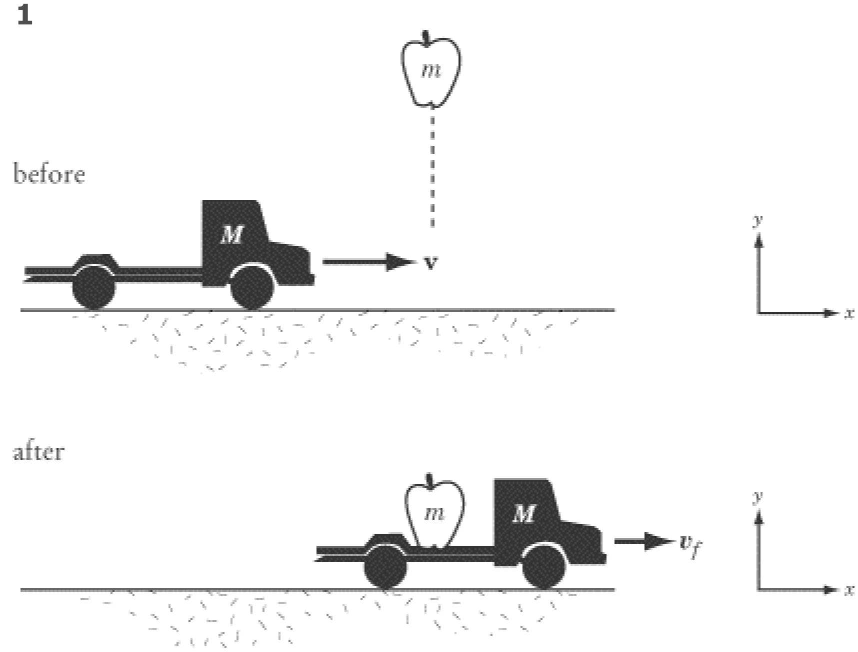
The conservation of momentum only applies to systems that have no external forces acting upon them. We call such a system a closed or **isolated system**: objects within the system may exert forces on other objects within the system (e.g., the cue ball can exert a force on the eight ball and vice versa), but no force can be exerted between an object outside the system and an object within the system. As a result, conservation of momentum does not apply to systems where friction is a factor.

Conservation of Momentum on SAT II Physics

The conservation of momentum may be tested both quantitatively and qualitatively on SAT II Physics. It is quite possible, for instance, that SAT II Physics will contain a question or two that involves a calculation based on the law of conservation of momentum. In such a question, "conservation of momentum" will not be mentioned explicitly, and even "momentum" might not be mentioned. Most likely, you will be asked to calculate the velocity of a moving object after a collision of some sort, a calculation that demands that you apply the law of conservation of momentum.

Alternately, you may be asked a question that simply demands that you identify the law of conservation of momentum and know how it is applied. The first example we will look at is of this qualitative type, and the second example is of a quantitative conservation of momentum question.

EXAMPLE 1



An apple of mass *m* falls into the bed of a moving toy truck of mass *M*. Before the apple lands in the car, the car is moving at constant velocity ***v*** on a frictionless track. Which of the following laws would you use to find the speed of the toy truck after the apple has landed?

(A) Newton's First Law

(B) Newton's Second Law

(C) Kinematic equations for constant acceleration

(D) Conservation of mechanical energy

(E) Conservation of linear momentum

Although the title of the section probably gave the solution away, we phrase the problem in this way because you'll find questions of this sort quite a lot on SAT II Physics. You can tell a question will rely on the law of conservation of momentum for its solution if you are given the initial velocity of an object and are asked to determine its final velocity after a change in mass or a collision with another object.

Some Supplemental Calculations

But how would we use conservation of momentum to find the speed of the toy truck after the apple has landed?

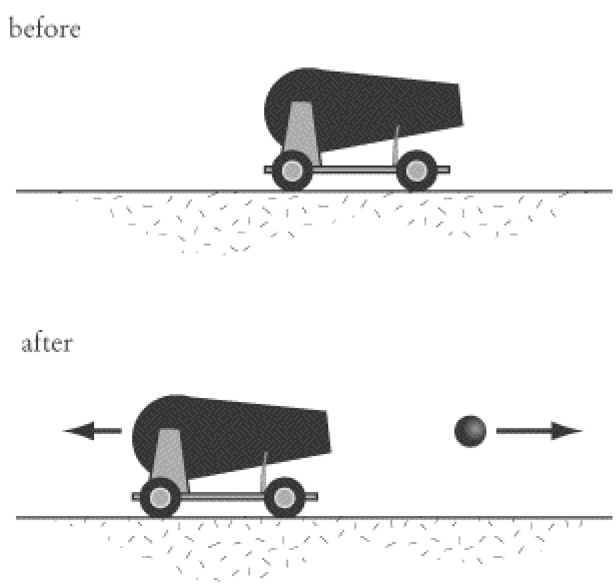
First, note that the net force acting in the x direction upon the apple and the toy truck is zero. Consequently, linear momentum in the x direction is conserved. The initial momentum of the system in the x direction is the momentum of the toy truck, $p_i = Mv$.

Once the apple is in the truck, both the apple and the truck are traveling at the same speed, v_f. Therefore, $p_f = mv_f + Mv_f = (m + M)v_f$. Equating p_i and p_f, we find:

$$Mv = (m + M)v_f$$
$$v_f = \frac{Mv}{m + M}$$

As we might expect, the final velocity of the toy truck is less than its initial velocity. As the toy truck gains the apple as cargo, its mass increases and it slows down. Because momentum is conserved and is directly proportional to mass and velocity, any increase in mass must be accompanied by a corresponding decrease in velocity.

EXAMPLE 2



A cannon of mass 1000 kg launches a cannonball of mass 10 kg at a velocity of 100 m/s. At what speed does the cannon recoil?

Questions involving firearms recoil are a common way in which SAT II Physics may test your knowledge of conservation of momentum. Before we dive into the math, let's get a clear picture of what's going on here. Initially the cannon and cannonball are at rest, so the total momentum of the system is zero. No external forces act on the system in the horizontal direction, so the system's linear momentum in this direction is constant. Therefore the momentum of the system both before and after the cannon fires must be zero.

Now let's make some calculations. When the cannon is fired, the cannonball shoots forward with momentum (10 kg)(100 m/s) = 1000 kg · m/s. To keep the total momentum of the system at zero, the cannon must then recoil with an equal momentum:

$$p_{\text{cannon}} = mv_{\text{cannon}}$$
$$1000 \text{ kg} \cdot \text{m/s} = (1000 \text{ kg})v_{\text{cannon}}$$
$$v_{\text{cannon}} = 1 \text{ m/s}$$

Any time a gun, cannon, or an artillery piece releases a projectile, it experiences a "kick" and moves in the opposite direction of the projectile. The more massive the firearm, the slower it moves.

Collisions

A **collision** occurs when two or more objects hit each other. When objects collide, each object feels a force for a short amount of time. This force imparts an impulse, or changes the momentum of each of the colliding objects. But if the system of particles is isolated, we know that momentum is conserved. Therefore, while the momentum of each individual particle involved in the collision changes, the total momentum of the system remains constant.

The procedure for analyzing a collision depends on whether the process is **elastic** or **inelastic**. Kinetic energy is conserved in elastic collisions, whereas kinetic energy is converted into other forms of energy during an inelastic collision. In both types of collisions, momentum is conserved.

Elastic Collisions

Anyone who plays pool has observed elastic collisions. In fact, perhaps you'd better head over to the pool hall right now and start studying! Some kinetic energy is converted into sound energy when pool balls collide—otherwise, the collision would be silent—and a very small amount of kinetic energy is lost to friction. However, the dissipated energy is such a small fraction of the ball's kinetic energy that we can treat the collision as elastic.

Equations for Kinetic Energy and Linear Momentum

Let's examine an elastic collision between two particles of mass m_1 and m_2, respectively. Assume that the collision is head-on, so we are dealing with only one dimension—you are unlikely to find two-dimensional collisions of any complexity on SAT II Physics. The velocities of the particles before the elastic collision are v_1 and v_2, respectively. The velocities of the particles after the elastic collision are v_1' and v_2'. Applying the law of conservation of kinetic energy, we find:

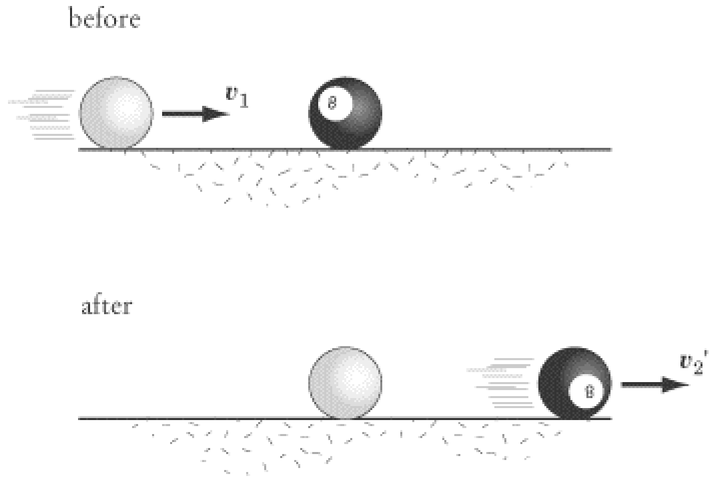
$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}m_1v_1'^2 + \frac{1}{2}m_2v_2'^2$$

Applying the law of conservation of linear momentum:

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2'$$

These two equations put together will help you solve any problem involving elastic collisions. Usually, you will be given quantities for m_1, m_2, v_1 and v_2, and can then manipulate the two equations to solve for v_1' and v_2'.

EXAMPLE



A pool player hits the eight ball, which is initially at rest, head-on with the cue ball. Both of these balls have the same mass, and the velocity of the cue ball is initially v_1. What are the velocities of the two balls after the collision? Assume the collision is perfectly elastic.

Substituting $m_1 = m_2 = m$ and $v_2 = 0$ into the equation for conservation of kinetic energy we find:

$$\frac{1}{2}mv_1^2 = \frac{1}{2}m(v_1'^2 + v_2'^2)$$
$$v_1^2 = v_1'^2 + v_2'^2$$

Applying the same substitutions to the equation for conservation of momentum, we find:

$$mv_1 = mv_1' + mv_2'$$
$$v_1 = v_1' + v_2'$$

If we square this second equation, we get:

$$v_1^2 = v_1'^2 + v_2'^2 + 2v_1'v_2'$$

By subtracting the equation for kinetic energy from this equation, we get:

$$2v_1'v_2' = 0$$

The only way to account for this result is to conclude that $v_1' = 0$ and consequently $v_1 = v_2'$. In plain English, the cue ball and the eight ball swap velocities: after the balls collide, the cue ball stops and the eight ball shoots forward with the initial velocity of the cue ball. This is the simplest form of an elastic collision, and also the most likely to be tested on SAT II Physics.

Inelastic Collisions

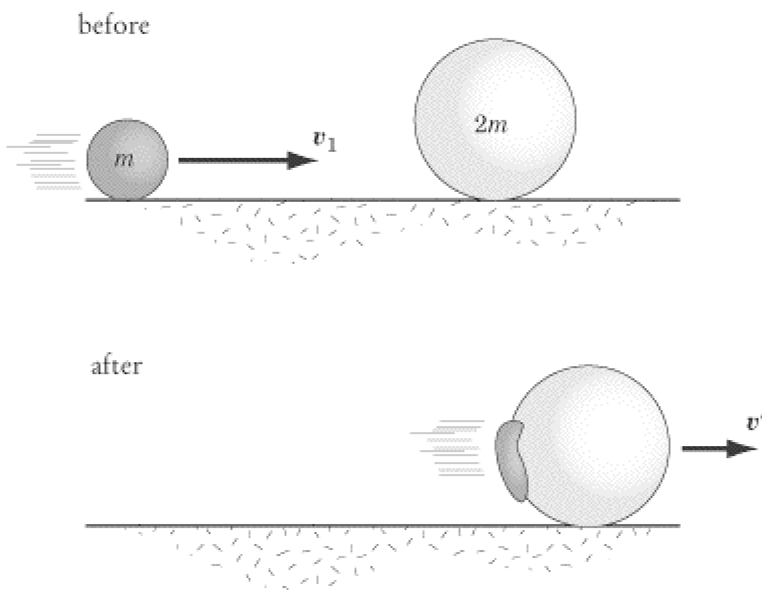
Most collisions are inelastic because kinetic energy is transferred to other forms of energy—such as thermal energy, potential energy, and sound—during the collision process. If you are asked to determine if a collision is elastic or inelastic, calculate the kinetic energy of the bodies before and after the collision. If kinetic energy is not conserved, then the collision is inelastic. Momentum is conserved in all inelastic collisions.

On the whole, inelastic collisions will only appear on SAT II Physics qualitatively. You may be asked to identify a collision as inelastic, but you won't be expected to calculate the resulting velocities of the objects involved in the collision. The one exception to this rule is in the case of **completely inelastic collisions**.

Completely Inelastic Collisions

A completely inelastic collision, also called a "perfectly" or "totally" inelastic collision, is one in which the colliding objects stick together upon impact. As a result, the velocity of the two colliding objects is the same after they collide. Because $v_1' = v_2' = v'$, it is possible to solve problems asking about the resulting velocities of objects in a completely inelastic collision using only the law of conservation of momentum.

EXAMPLE



Two gumballs, of mass *m* and mass 2*m* respectively, collide head-on. Before impact, the gumball of mass *m* is moving with a velocity v_1, and the gumball of mass 2*m* is stationary. What is the final velocity, v', of the gumball wad?

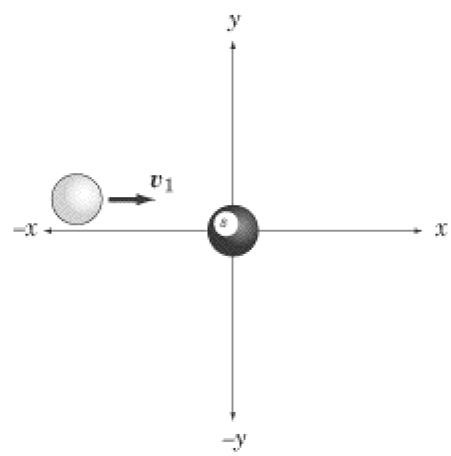
First, note that the gumball wad has a mass of $m + 2m = 3m$. The law of conservation of momentum tells us that $mv_1 = 3mv'$, and so $v' = v_1/3$. Therefore, the final gumball wad moves in the same direction as the first gumball, but with one-third of its velocity.

Collisions in Two Dimensions

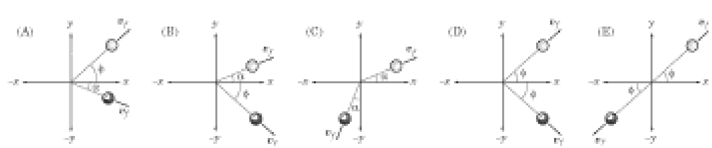
Two-dimensional collisions, while a little more involved than the one-dimensional examples we've looked at so far, can be treated in exactly the same way as their one-dimensional counterparts. Momentum is still conserved, as is kinetic energy in the case of elastic collisions. The significant difference is that you will have to break the trajectories of objects down into *x*- and *y*-components. You will then be able to deal with the two components separately: momentum is conserved in the *x* direction, and momentum is conserved in the *y* direction. Solving a problem of two-dimensional collision is effectively the same thing as solving two problems of one-dimensional collision.

Because SAT II Physics generally steers clear of making you do too much math, it's unlikely that you'll be faced with a problem where you need to calculate the final velocities of two objects that collide two-dimensionally. However, questions that test your understanding of two-dimensional collisions qualitatively are perfectly fair game.

EXAMPLE



A pool player hits the eight ball with the cue ball, as illustrated above. Both of the billiard balls have the same mass, and the eight ball is initially at rest. Which of the figures below illustrates a possible trajectory of the balls, given that the collision is elastic and both balls move at the same speed?

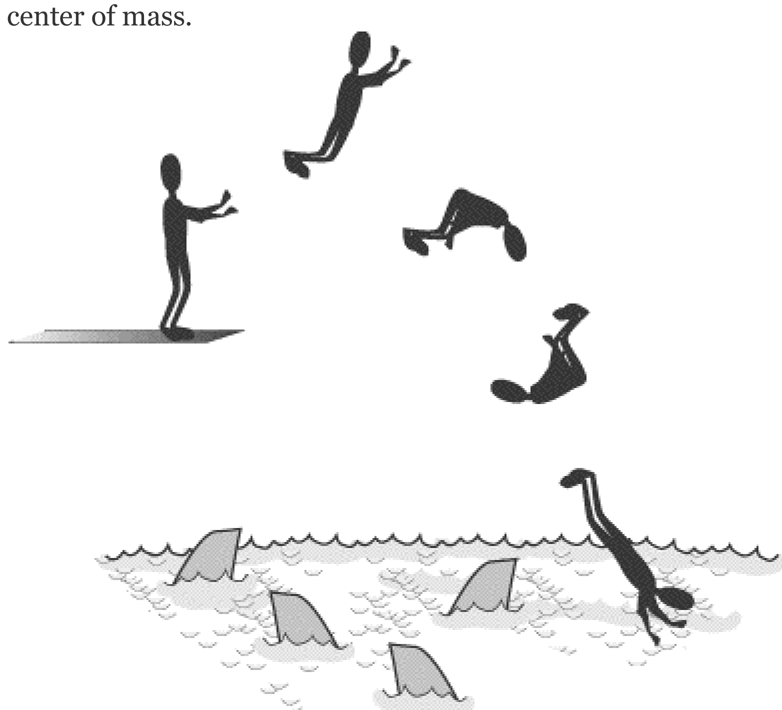


The correct answer choice is **D**, because momentum is not conserved in any of the other figures. Note that the initial momentum in the *y* direction is zero, so the momentum of the balls in the *y* direction after the collision must also be zero. This is only true for choices **D** and **E**. We also know that the initial momentum in the *x* direction is positive, so the final momentum in the *x* direction must also be positive, which is not true for **E**.

Center of Mass

When calculating trajectories and collisions, it's convenient to treat extended bodies, such as boxes and balls, as point masses. That way, we don't need to worry about the shape of an object, but can still take into account its mass and trajectory. This is basically what we do with free-body diagrams. We can treat objects, and even systems, as point masses, even if they have very strange shapes or are rotating in complex ways. We can make this simplification because there is always a point in the object or system that has the same trajectory as the object or system as a whole would have if all its mass were concentrated in that point. That point is called the object's or system's **center of mass**.

Consider the trajectory of a diver jumping into the water. The diver's trajectory can be broken down into the translational movement of his center of mass, and the rotation of the rest of his body about that center of mass.



A human being's center of mass is located somewhere around the pelvic area. We see here that, though the diver's head and feet and arms can rotate and move gracefully in space, the center of mass in his pelvic area follows the inevitable parabolic trajectory of a body moving under the influence of gravity. If we wanted to represent the diver as a point mass, this is the point we would choose.

Our example suggests that Newton's Second Law can be rewritten in terms of the motion of the center of mass:

$$\boldsymbol{F}_{\text{net}} = M\boldsymbol{a}_{\text{cm}}$$

Put in this form, the Second Law states that the net force acting on a system, $\boldsymbol{F}_{\text{net}}$, is equal to the product of the total mass of the system, M, and the acceleration of the center of mass, $\boldsymbol{a}_{\text{cm}}$. Note that if the net force acting on a system is zero, then the center of mass does not accelerate.

Similarly, the equation for linear momentum can be written in terms of the velocity of the center of mass:

$$p = Mv_{cm}$$

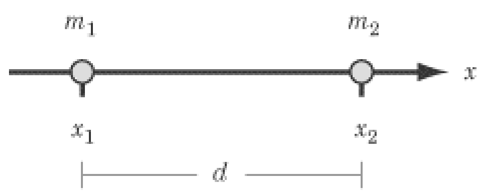
You will probably never need to plug numbers into these formulas for SAT II Physics, but it's important to understand the principle: the rules of dynamics and momentum apply to systems as a whole just as they do to bodies.

Calculating the Center of Mass

The center of mass of an object of uniform density is the body's geometric center. Note that the center of mass does not need to be located within the object itself. For example, the center of mass of a donut is in the center of its hole.

For a System of Two Particles

For a collection of particles, the center of mass can be found as follows. Consider two particles of mass m_1 and m_2 separated by a distance *d*:



If you choose a coordinate system such that both particles fall on the *x*-axis, the center of mass of this system, x_{cm}, is defined by:

$$x_{cm} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$

For a System in One Dimension

We can generalize this definition of the center of mass for a system of *n* particles on a line. Let the positions of these particles be x_1, x_2, . . ., x_n. To simplify our notation, let *M* be the total mass of all *n* particles in the system, meaning $M = m_1 + m_2 + \ldots + m_n$. Then, the center of mass is defined by:

$$x_{cm} = \frac{m_1x_1 + m_2x_2 + \cdots + m_nx_n}{M}$$

For a System in Two Dimensions

Defining the center of mass for a two-dimensional system is just a matter of reducing each particle in the system to its *x*- and *y*-components. Consider a system of *p*articles in a random arrangement of *x*-coordinates x_1, x_2, . . . , x_n and *y*-coordinates y_1, y_2, . . ., y_n. The *x*-coordinate of the center of mass is given in the equation above, while the *y*-coordinate of the center of mass is:

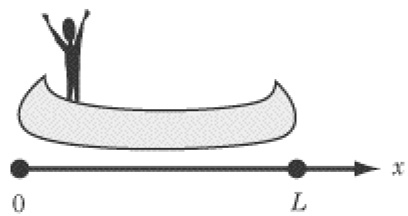
$$y_{cm} = \frac{m_1y_1 + m_2y_2 + \cdots + m_ny_n}{M}$$

How Systems Will Be Tested on SAT II Physics

The formulas we give here for systems in one and two dimensions are general formulas to help you understand the principle by which the center of mass is determined. Rest assured that for SAT II Physics, you'll never have to plug in numbers for mass and position for a system of several particles. However, your understanding of center of mass may be tested in less mathematically rigorous ways.

For instance, you may be shown a system of two or three particles and asked explicitly to determine the center of mass for the system, either mathematically or graphically. Another example, which we treat below, is that of a system consisting of two parts, where one part moves relative to the other. In this cases, it is important to remember that the center of mass of the system as a whole doesn't move.

EXAMPLE



A fisherman stands at the back of a perfectly symmetrical boat of length L. The boat is at rest in the middle of a perfectly still and peaceful lake, and the fisherman has a mass $^1/_4$ that of the boat. If the fisherman walks to the front of the boat, by how much is the boat displaced?

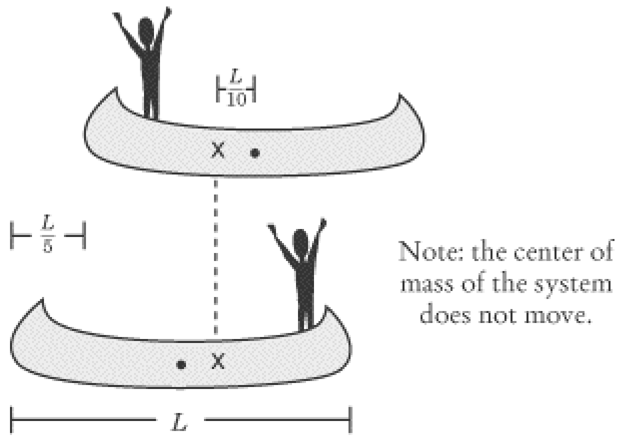
If you've ever tried to walk from one end of a small boat to the other, you may have noticed that the boat moves backward as you move forward. That's because there are no external forces acting on the system, so the system as a whole experiences no net force. If we recall the equation $\mathbf{F}_{net} = M\mathbf{a}_{cm}$, the center of mass of the system cannot move if there is no net force acting on the system. The fisherman can move, the boat can move, but the system as a whole must maintain the same center of mass. Thus, as the fisherman moves forward, the boat must move backward to compensate for his movement.

Because the boat is symmetrical, we know that the center of mass of the boat is at its geometrical center, at $x = {}^L/_2$. Bearing this in mind, we can calculate the center of mass of the system containing the fisherman and the boat:

$$x_{cm} = \frac{(\frac{m}{4})(0) + m\frac{L}{2}}{m + \frac{1}{4}m} = \frac{2}{5}L$$

Now let's calculate where the center of mass of the fisherman-boat system is relative to the boat after the fisherman has moved to the front. We know that the center of mass of the fisherman-boat system hasn't moved relative to the water, so its displacement with respect to the boat represents how much the boat has been displaced with respect to the water.

In the figure below, the center of mass of the boat is marked by a dot, while the center of mass of the fisherman-boat system is marked by an x.



At the front end of the boat, the fisherman is now at position L, so the center of mass of the fisherman-boat system relative to the boat is

$$x_{cm} = \frac{(\frac{m}{4})(L) + m\frac{L}{2}}{m + \frac{1}{4}m} = \frac{3}{5}L$$

The center of mass of the system is now $^3/_5$ from the back of the boat. But we know the center of mass hasn't moved, which means the boat has moved backward a distance of $^1/_5 L$, so that the point $^3/_5 L$ is now located where the point $^2/_5 L$ was before the fisherman began to move.

Key Formulas

Linear Momentum	$\mathbf{p} = m\mathbf{v}$
Impulse of a Constant Force	$\mathbf{J} = \mathbf{F}\Delta t = \Delta\mathbf{p}$
Conservation of Energy for an Elastic Collision of Two Particles	$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}m_1v_1'^2 + \frac{1}{2}m_2v_2'^2$
Conservation of Momentum for a Collision of Two Particles	$m_1v_1 + m_2v_2 = m_1v_1' + m_2v_2'$
Center of Mass for a System of particles	$x_{cm} = \frac{m_1x_1 + m_2x_2 + \cdots + m_nx_n}{M}$ $y_{cm} = \frac{m_1y_1 + m_2y_2 + \cdots + m_ny_n}{M}$

Acceleration of the Center of Mass	$\boldsymbol{F}_{\text{net}} = M\boldsymbol{a}_{\text{cm}}$
Momentum of the Center of Mass	$\boldsymbol{p} = M\boldsymbol{v}_{\text{cm}}$

Practice Questions

1. An athlete of mass 70.0 kg applies a force of 500 N to a 30.0 kg luge, which is initially at rest, over a period of 5.00 s before jumping onto the luge. Assuming there is no friction between the luge and the track on which it runs, what is its velocity after the athlete jumps on?

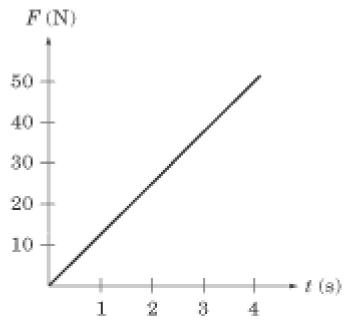
   (A) 12.5 m/s

   (B) 25.0 m/s

   (C) 35.7 m/s

   (D) 83.3 m/s

   (E) 100 m/s



2. The graph above shows the amount of force applied to an initially stationary 20 kg curling rock over time. What is the velocity of the rock after the force has been applied to it?

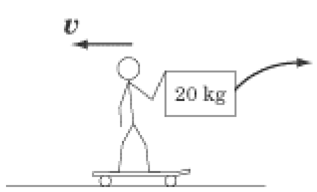
   (A) 1.25 m/s

   (B) 5 m/s

   (C) 10 m/s

   (D) 25 m/s

   (E) 50 m/s



3. A 60 kg man holding a 20 kg box rides on a skateboard at a speed of 7 m/s. He throws the box behind him, giving it a velocity of 5 m/s. with respect to the ground. What is his velocity after throwing the object?

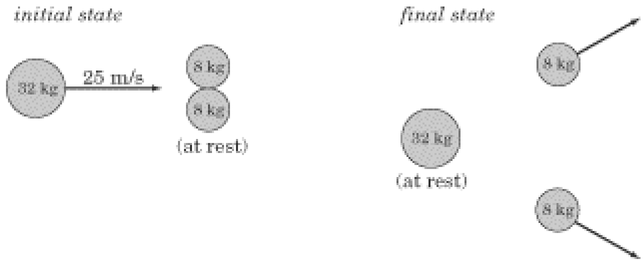
   (A) 8 m/s

   (B) 9 m/s

   (C) 10 m/s

(D) 11 m/s

(E) 12 m/s



4. A scattering experiment is done with a 32 kg disc and two 8 kg discs on a frictionless surface. In the initial state of the experiment, the heavier disc moves in the *x* direction with velocity *v* = 25 m/s toward the lighter discs, which are at rest. The discs collide elastically. In the final state, the heavy disc is at rest and the two smaller discs scatter outward with the same speed. What is the *x*-component of the velocity of each of the 8 kg discs in the final state?

(A) 12.5 m/s

(B) 16 m/s

(C) 25 m/s

(D) 50 m/s

(E) 100 m/s

5. An moving object has kinetic energy *KE* = 100 J and momentum *p* = 50 kg Â· m/s. What is its mass?

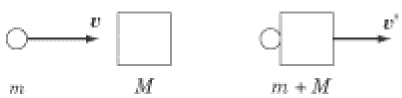
(A) 2 kg

(B) 4 kg

(C) 6.25 kg

(D) 12.5 kg

(E) 25 kg



6. An object of mass *m* moving with a velocity *v* collides with another object of mass *M*. If the two objects stick together, what is their velocity?

(A) $\frac{M}{m + M}v$

(B) $\frac{m}{m + M}v$

(C) $\frac{m + M}{m}v$

(D) $\frac{m + M}{M}v$

(E) Zero

7. A body of mass *m* sliding along a frictionless surface collides with another body of mass *m*, which is stationary before impact. The two bodies stick together. If the kinetic energy of the two-body system is *E*, what is the initial velocity of the first mass before impact?

(A) $\sqrt{E/2m}$

(B) $\sqrt{2E/2m}$

(C) $\sqrt{2E/m}$

(D) $\sqrt{E/m}$

(E) $2\sqrt{E/m}$

8. A hockey puck of mass *m* is initially at rest on a frictionless ice rink. A player comes and hits the puck, imparting an impulse of *J*. If the puck then collides with another object of mass *M* at rest and sticks to it, what is the final velocity of the two-body system?

(A) $\frac{J}{m}$

(B) $\frac{J}{M}$

(C) $\frac{J}{m+M}$

(D) $\frac{m+M}{J}$

(E) $\frac{M}{J}$

Questions 9 and 10 refer to two 1 kg masses moving toward each other, one mass with velocity v_1 = 10 m/s, the other with velocity v_2 = 20 m/s.

9. What is the velocity of the center of mass?

(A) 0 m/s

(B) 5 m/s to the left

(C) 10 m/s to the left

(D) 15 m/s to the left

(E) 20 m/s to the left

10. What is the total energy of the system?

(A) 50 J

(B) 150 J

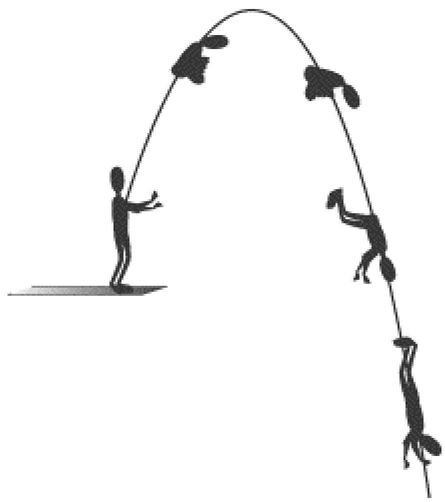
(C) 200 J

(D) 250 J

(E) 400 J

Rotational Motion

UNTIL THIS CHAPTER, WE HAVE FOCUSED almost entirely on **translational motion**, the motion of bodies moving through space. But there is a second kind of motion, called **rotational motion**, which deals with the rotation of a body about its center of mass. The movement of any object can be described through the combination of translational motion of the object's center of mass and its rotational motion about that center of mass. For example, look at the diver jumping into the water that we saw in the previous chapter.



The diver's translational motion is the parabolic trajectory of her center of mass. However, if that were the only motion of the diver's body, diving competitions would be considerably more boring. What astonishes fans and impresses judges is the grace and fluidity of the rotational motion of the diver's arms, legs, feet, etc., about that center of mass.

You will find that rotational motion and translational motion have a lot in common. In fact, aside from a few basic differences, the mechanics of rotational motion are identical to those of translational motion. We'll begin this chapter by introducing some basic concepts that are distinct to rotational motion. After that, we will recapitulate what we covered in the chapters on translational motion, explaining how the particularities of rotational motion differ from their translational counterparts. We will examine, in turn, the rotational equivalents for kinematic motion, dynamics, energy, and momentum.
There will be at most one or two questions on rotational motion on any given SAT II test. On the whole, they tend to center around the concepts of torque and equilibrium.

Important Definitions

There are a few basic physical concepts that are fundamental to a proper understanding of rotational motion. With a steady grasp of these concepts, you should encounter no major difficulties in making the transition between the mechanics of translational motion and of rotational motion.

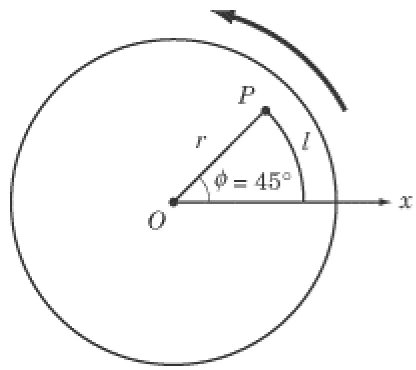
Rigid Bodies

The questions on rotational motion on SAT II Physics deal only with **rigid bodies**. A rigid body is an object that retains its overall shape, meaning that the particles that make up the rigid body stay in the same position relative to one another. A pool ball is one example of a rigid body since the shape of the ball is constant as it rolls and spins. A wheel, a record, and a top are other examples of rigid bodies that commonly appear in questions involving rotational motion. By contrast, a slinky is not a rigid body, because its coils expand, contract, and bend, so that its motion would be considerably more difficult to predict if you were to spin it about.

Center of Mass

The **center of mass** of an object, in case you have forgotten, is the point about which all the matter in the object is evenly distributed. A net force acting on the object will accelerate it in just the same way as if all the mass of the object were concentrated in its center of mass. We looked at the concept of center of mass in the previous chapter's discussion of linear momentum. The concept of center of mass will play an even more central role in this chapter, as rotational motion is essentially defined as the rotation of a body about its center of mass.

Axis of Rotation

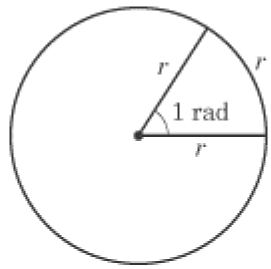


The rotational motion of a rigid body occurs when every point in the body moves in a circular path around a line called the **axis of rotation**, which cuts through the center of mass. One familiar example of rotational motion is that of a spinning wheel. In the figure at right, we see a wheel rotating counterclockwise around an axis labeled O that is perpendicular to the page. As the wheel rotates, every point in the rigid body makes a circle around the axis of rotation, O.

Radians

We're all very used to measuring angles in degrees, and know perfectly well that there are 360° in a circle, 90° in a right angle, and so on. You've probably noticed that 360 is also a convenient number because so many other numbers divide into it. However, this is a totally arbitrary system that has its origins in the Ancient Egyptian calendar which was based on a 360-day year.

It makes far more mathematical sense to measure angles in **radians** (rad). If we were to measure the arc of a circle that has the same length as the radius of that circle, then one radian would be the angle made by two radii drawn to either end of the arc.



Converting between Degrees and Radians

It is unlikely that SAT II Physics will specifically ask you to convert between degrees and radians, but it will save you time and headaches if you can make this conversion quickly and easily. Just remember this formula:

$$y^\circ = \frac{180x}{\pi}\text{rad}$$

You'll quickly get used to working in radians, but below is a conversion table for the more commonly occurring angles.

Value in degrees	Value in radians
30	$\pi/6$
45	$\pi/4$
60	$\pi/3$
90	$\pi/2$
180	π
360	2π

Calculating the Length of an Arc

The advantage of using radians instead of degrees, as will quickly become apparent, is that the radian is based on the nature of angles and circles themselves, rather than on the arbitrary fact of how long it takes our Earth to circle the sun.

For example, calculating the length of any arc in a circle is much easier with radians than with degrees. We know that the circumference of a circle is given by $P = 2\pi r$, and we know that there are 2π radians in a circle. If we wanted to know the length, l, of the arc described by any angle θ, we would know that this arc is a fraction of the perimeter, $(\theta/2\pi)P$. Because $P = 2\pi r$, the length of the arc would be:

$$l = \frac{\theta}{2\pi}P = \frac{\theta}{2\pi}2\pi r = \theta r$$

Rotational Kinematics

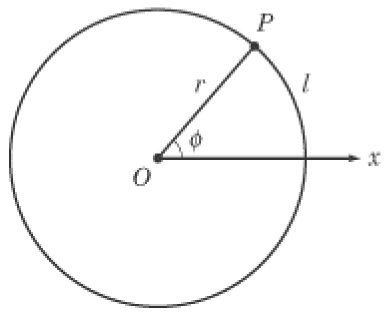
You are now going to fall in love with the word angular. You'll find that for every term in kinematics that you're familiar with, there's an "angular" counterpart: **angular displacement**, **angular velocity**, **angular acceleration**, etc. And you'll find that, "angular" aside, very little changes when dealing with rotational kinematics.

Angular Position, Displacement, Velocity, and Acceleration

SAT II Physics is unlikely to have any questions that simply ask you to calculate the angular position, displacement, velocity, or acceleration of a rotating body. However, these concepts form the basis of rotational mechanics, and the questions you *will* encounter on SAT II Physics will certainly be easier if you're familiar with these fundamentals.

Angular Position

By convention, we measure angles in a circle in a counterclockwise direction from the positive *x*-axis. The **angular position** of a particle is the angle, ϕ, made between the line connecting that particle to the origin, O, and the positive *x*-axis, measured counterclockwise. Let's take the example of a point P on a rotating wheel:



In this figure, point P has an angular position of ϕ. Note that every point on the line $\overline{OP}$ has the same angular position: the angular position of a point does not depend on how far that point is from the origin, O.

We can relate the angular position of P to the length of the arc of the circle between P and the *x*-axis by means of an easy equation:

$$\phi = \frac{l}{r}$$

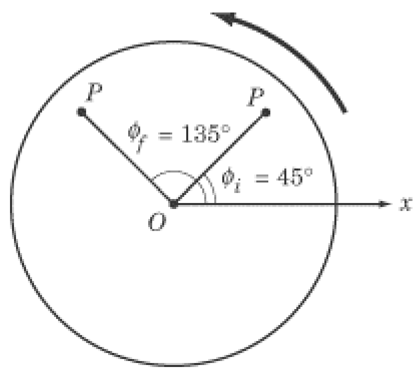
In this equation, l is the length of the arc, and r is the radius of the circle.

Angular Displacement

Now imagine that the wheel is rotated so that every point on line $\overline{OP}$ moves from an initial angular position of ϕ_i to a final angular position of ϕ_f. The **angular displacement**, θ, of line $\overline{OP}$ is:

$$\theta = \phi_f - \phi_i$$

For example, if you rotate a wheel counterclockwise such that the angular position of line $\overline{OP}$ changes from ϕ_i = 45° = π/4 to ϕ_f = 135° = 3π/4, as illustrated below, then the angular displacement of line $\overline{OP}$ is 90° or π/2 radians.



For line $\overline{OP}$ to move in the way described above, every point along the line must rotate 90° counterclockwise. By definition, the particles that make up a rigid body must stay in the same relative position to one another. As a result, the angular displacement is the same for every point in a rotating rigid body.

Also note that the angular distance a point has rotated may or may not equal that point's angular displacement. For example, if you rotate a record 45° clockwise and then 20° counterclockwise, the angular displacement of the record is 25°, although the particles have traveled a total angular distance of 65°. Hopefully, you've already had it hammered into your head that distance and displacement are not the same thing: well, the same distinction applies with angular distance and angular displacement.

Angular Velocity

Angular velocity, ω, is defined as the change in the angular displacement over time. Average angular velocity, $\bar{\omega}$, is defined by:

$$\bar{\omega} = \frac{\Delta\theta}{\Delta t}$$

Angular velocity is typically given in units of rad/s. As with angular displacement, the angular velocity of every point on a rotating object is identical.

Angular Acceleration

Angular acceleration, α, is defined as the rate of change of angular velocity over time. Average angular acceleration, $\bar{\alpha}$, is defined by:

$$\bar{\alpha} = \frac{\Delta\omega}{\Delta t}$$

Angular acceleration is typically given in units of rad/s^2.

Frequency and Period

You've encountered frequency and period when dealing with springs and simple harmonic motion, and you will encounter them again in the chapter on waves. These terms are also relevant to rotational motion, and SAT II Physics has been known to test the relation between angular velocity and angular frequency and period.

Angular Frequency

Angular frequency, f, is defined as the number of circular revolutions in a given time interval. It is commonly measured in units of Hertz (Hz), where 1 Hz = 1 s^{-1}. For example, the second hand on a clock completes one revolution every 60 seconds and therefore has an angular frequency of $^1/_{60}$ Hz.

The relationship between frequency and angular velocity is:

$$f = \frac{\omega}{2\pi}$$

For example, the second hand of a clock has an angular velocity of $\omega = \Delta\theta/\Delta t = 2\pi/60$ s. Plugging that value into the equation above, we get

$$f = \frac{2\pi/(60\text{ s})}{2\pi} = \frac{1}{60\text{ s}} = 1/60\text{ Hz}$$

which we already determined to be the frequency of the second hand of a clock.

Angular Period

Angular period, T, is defined as the time required to complete one revolution and is related to frequency by the equation:

$$T = \frac{1}{f}$$

Since we know that the frequency of the second hand is $^1/_{60}$ Hz, we can quickly see that the period of the second hand is 60 s. It takes 60 seconds for the second hand to complete a revolution, so the period of the second hand is 60 seconds. Period and angular velocity are related by the equation

$$T = \frac{2\pi}{\omega}$$

EXAMPLE

> The Earth makes a complete rotation around the sun once every 365.25 days. What is the Earth's angular velocity?

The question tells us that the Earth has a period of $T = 365.25$ days. If we plug this value into the equation relating period and angular velocity, we find:

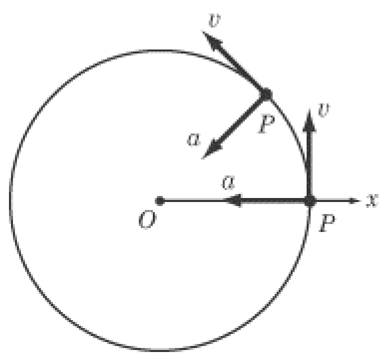
$$\omega = \frac{2\pi}{T} = \frac{2\pi}{365.25 \text{ days}}$$
$$= 1.7202 \times 10^{-2} \text{ rad/day}$$

Note, however, that this equation only gives us the Earth's angular velocity in terms of radians per day. In terms of radians per second, the correct answer is:

$$1.7202 \times 10^{-2} \text{ rad/day}\left(\frac{1 \text{ day}}{8.64 \times 10^4 \text{ s}}\right) = 1.9910 \times 10^{-7} \text{ rad/s}$$

Relation of Angular Variables to Linear Variables

At any given moment, a rotating particle has an instantaneous linear velocity and an instantaneous linear acceleration. For instance, a particle P that is rotating counterclockwise will have an instantaneous velocity in the positive y direction at the moment it is at the positive x-axis. In general, a rotating particle has an instantaneous velocity that is tangent to the circle described by its rotation and an instantaneous acceleration that points toward the center of the circle.



On SAT II Physics, you may be called upon to determine a particle's linear velocity or acceleration given its angular velocity or acceleration, or vice versa. Let's take a look at how this is done.

Distance

We saw earlier that the angular position, ϕ, of a rotating particle is related to the length of the arc, l, between the particle's present position and the positive x-axis by the equation $\phi = l/r$, or $l = \phi r$. Similarly, for any angular displacement, θ, we can say that the length, l, of the arc made by a particle undergoing that displacement is

$$l = \theta r$$

Note that the length of the arc gives us a particle's distance traveled rather than its displacement, since displacement is a vector quantity measuring only the straight-line distance between two points, and not the length of the route traveled between those two points.

Velocity and Acceleration

Given the relationship we have determined between arc distance traveled, l, and angular displacement, θ, we can now find expressions to relate linear and angular velocity and acceleration.

We can express the instantaneous linear velocity of a rotating particle as $v = l/t$, where l is the distance traveled along the arc. From this formula, we can derive a formula relating linear and angular velocity:

$$\begin{aligned} v &= \frac{l}{t} = \frac{\theta r}{t} = \frac{\theta}{t} r \\ &= \omega r \end{aligned}$$

In turn, we can express linear acceleration as $a = v/t$, giving us this formula relating linear and angular acceleration:

$$\begin{aligned} a &= \frac{v}{t} = \frac{\omega r}{t} = \frac{\omega}{t} r \\ &= \alpha r \end{aligned}$$

EXAMPLE

> The radius of the Earth is approximately 6.4×10^6 m. What is the instantaneous velocity of a point on the surface of the Earth at the equator?

We know that the period of the Earth's rotation is 24 hours, or 8.64×10^4 seconds. From the equation relating period, T, to angular velocity, ω, we can find the angular velocity of the Earth:

$$\begin{aligned} T &= \frac{2\pi}{\omega} \\ \omega &= \frac{2\pi}{T} = \frac{2\pi}{8.64 \times 10^4 \text{ s}} \\ &= 7.27 \times 10^{-5} \text{ rad/s} \end{aligned}$$

Now that we know the Earth's angular velocity, we simply plug that value into the equation for linear velocity:

$$\begin{aligned} v &= \omega r = (7.27 \times 10^{-5} \text{rad/s})(6.4 \times 10^6 \text{m}) \\ &= 4.7 \times 10^2 \text{m/s} \end{aligned}$$

They may not notice it, but people living at the equator are moving faster than the speed of sound.

Equations of Rotational Kinematics

In Chapter 2 we defined the kinematic equations for bodies moving at constant acceleration. As we have seen, there are very clear rotational counterparts for linear displacement, velocity, and acceleration, so we are able to develop an analogous set of five equations for solving problems in rotational kinematics:

$$\phi = \phi_0 + \frac{1}{2}(\omega + \omega_0)t$$
$$\omega = \omega_0 + \alpha t$$
$$\phi = \phi_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$
$$\phi = \phi_0 + \omega t - \frac{1}{2}\alpha t^2$$
$$\omega^2 = \omega_0^2 + 2\alpha(\phi - \phi_0)$$

In these equations, ω_0 is the object's initial angular velocity at its initial position, ϕ_0.

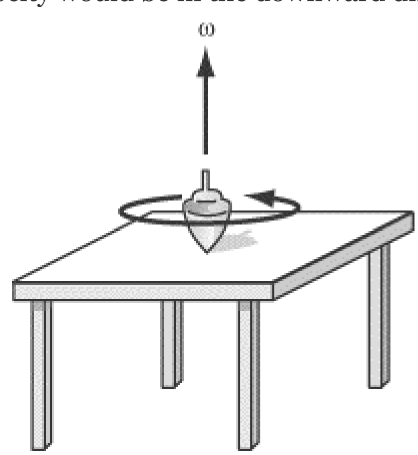
Any questions on SAT II Physics that call upon your knowledge of the kinematic equations will almost certainly be of the translational variety. However, it's worth noting just how deep the parallels between translational and rotational kinematics run.

Vector Notation of Rotational Variables

Angular velocity and angular acceleration are vector quantities; the equations above define their magnitudes but not their directions. Given that objects with angular velocity or acceleration are moving in a circle, how do we determine the direction of the vector? It may seem strange, but the direction of the vector for angular velocity or acceleration is actually perpendicular to the plane in which the object is rotating.

We determine the direction of the angular velocity vector using the **right-hand rule**. Take your right hand and curl your fingers along the path of the rotating particle or body. Your thumb then points in the direction of the angular velocity of the body. Note that the angular velocity is along the body's axis of rotation.

The figure below illustrates a top spinning counterclockwise on a table. The right-hand rule shows that its angular velocity is in the upward direction. Note that if the top were rotating clockwise, then its angular velocity would be in the downward direction.



To find the direction of a rigid body's angular acceleration, you must first find the direction of the body's angular velocity. Then, if the magnitude of the angular velocity is increasing, the angular acceleration is in the same direction as the angular velocity vector. On the other hand, if the magnitude of the angular velocity is decreasing, then the angular acceleration points in the direction opposite the angular velocity vector.

Rotational Dynamics

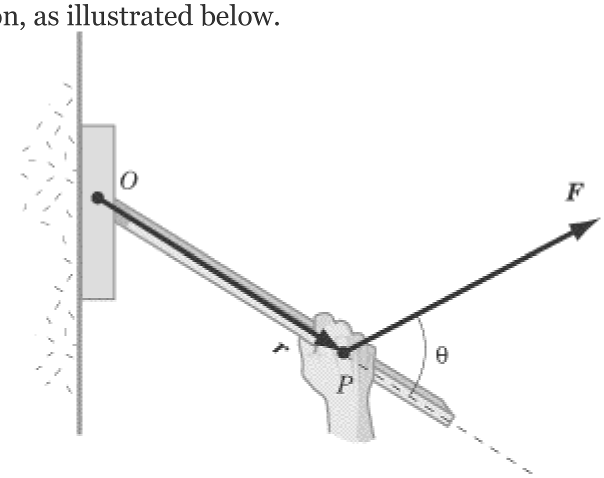
Just as we have rotational counterparts for displacement, velocity, and acceleration, so do we have rotational counterparts for force, mass, and Newton's Laws. As with angular kinematics, the key here is to recognize the striking similarity between rotational and linear dynamics, and to learn to move between the two quickly and easily.

Torque

If a net force is applied to an object's center of mass, it will not cause the object to rotate. However, if a net force is applied to a point other than the center of mass, it will affect the object's rotation. Physicists call the effect of force on rotational motion **torque**.

Torque Defined

Consider a lever mounted on a wall so that the lever is free to move around an axis of rotation O. In order to lift the lever, you apply a force $\mathbf{F}$ to point P, which is a distance r away from the axis of rotation, as illustrated below.



Suppose the lever is very heavy and resists your efforts to lift it. If you want to put all you can into lifting this lever, what should you do? Simple intuition would suggest, first of all, that you should lift with all your strength. Second, you should grab onto the end of the lever, and not a point near its axis of rotation. Third, you should lift in a direction that is perpendicular to the lever: if you pull very hard away from the wall or push very hard toward the wall, the lever won't rotate at all.

Let's summarize. In order to maximize torque, you need to:

1. Maximize the magnitude of the force, $\mathbf{F}$, that you apply to the lever.
2. Maximize the distance, r, from the axis of rotation of the point on the lever to which you apply the force.
3. Apply the force in a direction perpendicular to the lever.

We can apply these three requirements to an equation for torque, τ:

$$\tau = Fr\sin\theta$$

In this equation, θ is the angle made between the vector for the applied force and the lever.

Torque Defined in Terms of Perpendicular Components

There's another way of thinking about torque that may be a bit more intuitive than the definition provided above. Torque is the product of the distance of the applied force from the axis of rotation and the component of the applied force that is perpendicular to the lever arm. Or, alternatively, torque is the product of the applied force and the component of the length of the lever arm that runs perpendicular to the applied force.
We can express these relations mathematically as follows:

$$\tau = F_{\perp}r = Fr_{\perp}$$

where $F_{\perp}$ and $r_{\perp}$ are defined below.

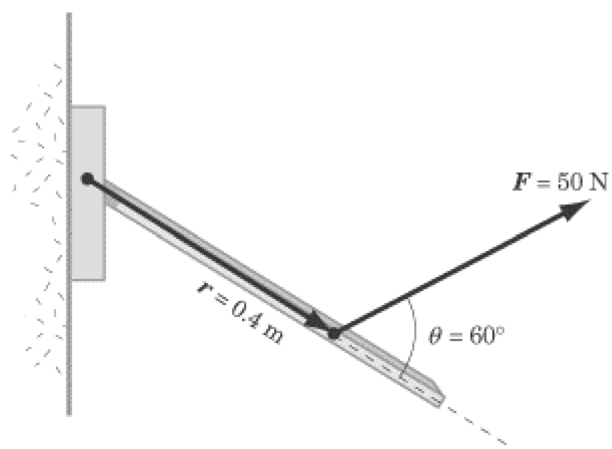
Torque Defined as a Vector Quantity

Torque, like angular velocity and angular acceleration, is a vector quantity. Most precisely, it is the cross product of the displacement vector, $\boldsymbol{r}$, from the axis of rotation to the point where the force is applied, and the vector for the applied force, $\boldsymbol{F}$.

$$\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F}$$

To determine the direction of the torque vector, use the right-hand rule, curling your fingers around from the $\boldsymbol{r}$ vector over to the $\boldsymbol{F}$ vector. In the example of lifting the lever, the torque would be represented by a vector at *O* pointing out of the page.

EXAMPLE



A student exerts a force of 50 N on a lever at a distance 0.4 m from its axis of rotation. The student pulls at an angle that is 60Â° above the lever arm. What is the torque experienced by the lever arm?

Let's plug these values into the first equation we saw for torque:

$$\tau = Fr\sin\theta = (50\text{ N})(0.4\text{ m})\sin 60^\circ$$
$$= 17.3\text{ N}\cdot\text{m}$$

This vector has its tail at the axis of rotation, and, according to the right-hand rule, points out of the page.

Newton's First Law and Equilibrium

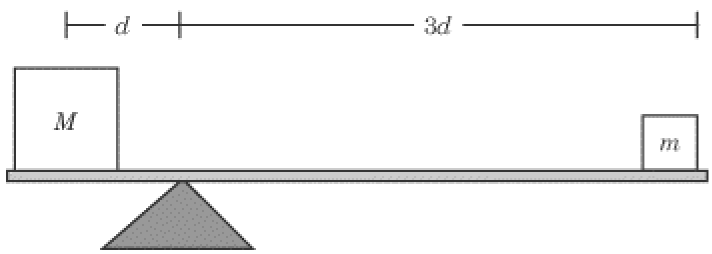
Newton's Laws apply to torque just as they apply to force. You will find that solving problems involving torque is made a great deal easier if you're familiar with how to apply Newton's Laws to them. The First Law states:

If the net torque acting on a rigid object is zero, it will rotate with a constant angular velocity.

The most significant application of Newton's First Law in this context is with regard to the concept of **equilibrium**. When the net torque acting on a rigid object is zero, and that object is not already rotating, it will not begin to rotate.

When SAT II Physics tests you on equilibrium, it will usually present you with a system where more than one torque is acting upon an object, and will tell you that the object is not rotating. That means that the net torque acting on the object is zero, so that the sum of all torques acting in the clockwise direction is equal to the sum of all torques acting in the counterclockwise direction. A typical SAT II Physics question will ask you to determine the magnitude of one or more forces acting on a given object that is in equilibrium.

EXAMPLE



Two masses are balanced on the scale pictured above. If the bar connecting the two masses is horizontal and mass less, what is the weight of mass *m* in terms of *M*?

Since the scale is not rotating, it is in equilibrium, and the net torque acting upon it must be zero. In other words, the torque exerted by mass M must be equal and opposite to the torque exerted by mass m. Mathematically,

$$\tau_M + \tau_m = 0$$
$$Mgd + (-mg(3d)) = 0$$
$$3mgd = Mgd$$
$$m = \frac{M}{3}$$

Because m is three times as far from the axis of rotation as M, it applies three times as much torque per mass. If the two masses are to balance one another out, then M must be three times as heavy as m.

Newton's Second Law

We have seen that acceleration has a rotational equivalent in angular acceleration, α, and that force has a rotational equivalent in torque, τ. Just as the familiar version of Newton's Second Law tells us that the acceleration of a body is proportional to the force applied to it, the rotational version of Newton's Second Law tells us that the angular acceleration of a body is proportional to the torque applied to it.

Of course, force is also proportional to mass, and there is also a rotational equivalent for mass: the **moment of inertia**, I, which represents an object's resistance to being rotated. Using the three variables, τ, I, and α, we can arrive at a rotational equivalent for Newton's Second Law:

$$\tau_{net} = I\alpha$$

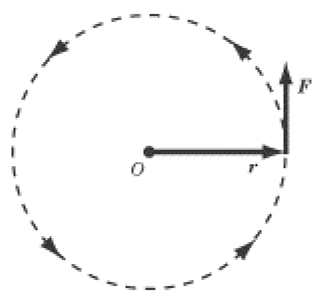
As you might have guessed, the real challenge involved in the rotational version of Newton's Second Law is sorting out the correct value for the moment of inertia.

Moment of Inertia

What might make a body more difficult to rotate? First of all, it will be difficult to set in a spin if it has a great mass: spinning a coin is a lot easier than spinning a lead block. Second, experience shows that the distribution of a body's mass has a great effect on its potential for rotation. In general, a body will rotate more easily if its mass is concentrated near the axis of rotation, but the calculations that go into determining the precise moment of inertia for different bodies is quite complex.

MOMENT OF INERTIA FOR A SINGLE PARTICLE

Consider a particle of mass m that is tethered by a mass less string of length r to point O, as pictured below:



The torque that produces the angular acceleration of the particle is $\tau = rF$, and is directed out of the page. From the linear version of Newton's Second Law, we know that $F = ma$ or $F = m\alpha r$. If we multiply both sides of this equation by r, we find:

$$\tau = mr^2\alpha$$

If we compare this equation to the rotational version of Newton's Second Law, we see that the moment of inertia of our particle must be mr^2.

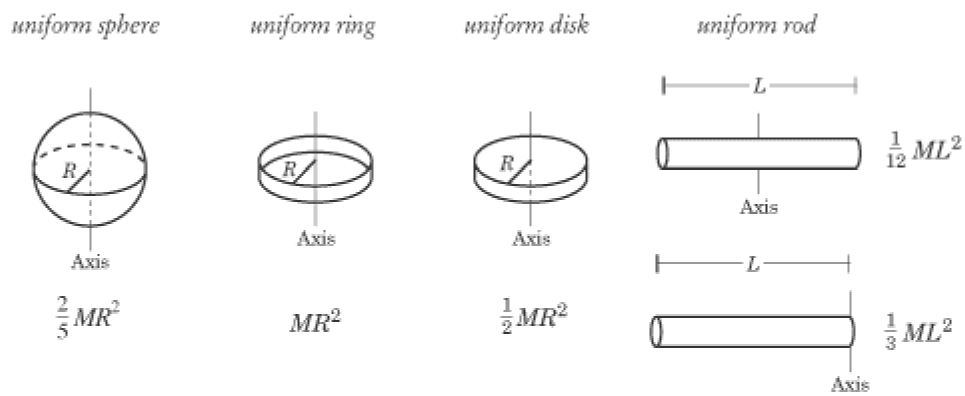
MOMENT OF INERTIA FOR RIGID BODIES

Consider a wheel, where every particle in the wheel moves around the axis of rotation. The net torque on the wheel is the sum of the torques exerted on each particle in the wheel. In its most general form, the rotational version of Newton's Second Law takes into account the moment of inertia of each individual particle in a rotating system:

$$\boldsymbol{\tau}_{\text{net}} = (\Sigma mr^2)\boldsymbol{\alpha}$$

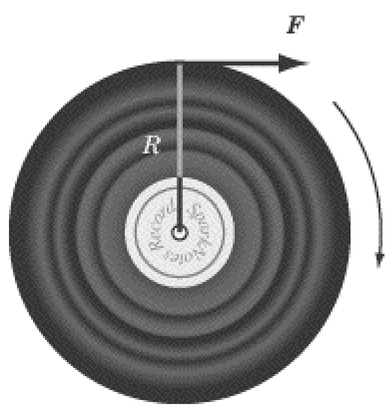
Of course, adding up the radius and mass of every particle in a system is very tiresome unless the system consists of only two or three particles. The moment of inertia for more complex systems can only be determined using calculus. SAT II Physics doesn't expect you to know calculus, so it will give you the moment of inertia for a complex body whenever the need arises. For your own reference, however, here is the moment of inertia for a few common shapes.

Moments of Inertia



In these figures, M is the mass of the rigid body, R is the radius of round bodies, and L is the distance on a rod between the axis of rotation and the end of the rod. Note that the moment of inertia depends on the shape and mass of the rigid body, as well as on its axis of rotation, and that for most objects, the moment of inertia is a multiple of MR^2.

EXAMPLE 1



A record of mass M and radius R is free to rotate around an axis through its center, O. A tangential force $\boldsymbol{F}$ is applied to the record. What must one do to maximize the angular acceleration?

(A) Make $\boldsymbol{F}$ and M as large as possible and R as small as possible

(B) Make M as large as possible and $\boldsymbol{F}$ and R as small as possible.

(C) Make $\boldsymbol{F}$ as large as possible and M and R as small as possible.

(D) Make R as large as possible and $\boldsymbol{F}$ and M as small as possible.

(E) Make $\boldsymbol{F}$, M, and R as large as possible.

To answer this question, you don't need to know exactly what a disc's moment of inertia is—you just need to be familiar with the general principle that it will be some multiple of MR^2.

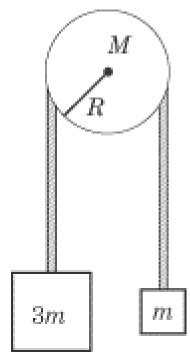
The rotational version of Newton's Second Law tells us that $\tau = I\alpha$, and so $\alpha = \boldsymbol{F}R/I$. Suppose we don't know what I is, but we know that it is some multiple of MR^2. That's enough to formulate an equation telling us all we need to know:

$$\alpha \propto \frac{FR}{MR^2} = \frac{F}{MR}$$

As we can see, the angular acceleration increases with greater force, and with less mass and radius; therefore **C** is the correct answer.

Alternately, you could have answered this question by physical intuition. You know that the more force you exert on a record, the greater its acceleration. Additionally, if you exert a force on a small, light record, it will accelerate faster than a large, massive record.

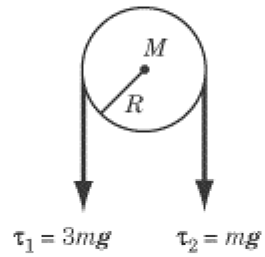
EXAMPLE 2



The masses in the figure above are initially held at rest and are then released. If the mass of the pulley is M, what is the angular acceleration of the pulley? The moment of inertia of a disk spinning around its center is $\frac{1}{2}MR^2$.

This is the only situation on SAT II Physics where you may encounter a pulley that is not considered mass less. Usually you can ignore the mass of the pulley block, but it matters when your knowledge of rotational motion is being tested.

In order to solve this problem, we first need to determine the net torque acting on the pulley, and then use Newton's Second Law to determine the pulley's angular acceleration. The weight of each mass is transferred to the tension in the rope, and the two forces of tension on the pulley block exert torques in opposite directions as illustrated below:



To calculate the torque one must take into account the tension in the ropes, the inertial resistance to motion of the hanging masses, and the inertial resistance of the pulley itself. The sum of the torques is given by:

$$\Sigma\tau = T_1R - T_2R = \frac{1}{2}MR^2\alpha$$

Solve for the tensions using Newton's second law. For Mass 1:

$$\Sigma F = 3mg - T_1 = 3ma$$

For Mass 2:

$$\Sigma F = mg - T_2 = (-ma)$$

Remember that $a = R\alpha$. Substitute into the first equation:

$$3R(mg - mR\alpha) - R(mg + mR\alpha) = \frac{1}{2}MR^2\alpha$$

$$2mgR - 4m\alpha R^2 = \frac{1}{2}MR^2\alpha$$

$$\alpha = \frac{2mg}{\frac{1}{2}MR + 4MR}$$

Because α is positive, we know that the pulley will spin in the counterclockwise direction and the 3*m* block will drop.

Kinetic Energy

There is a certain amount of energy associated with the rotational motion of a body, so that a ball rolling down a hill does not accelerate in quite the same way as a block sliding down a frictionless slope. Fortunately, the formula for rotational kinetic energy, much like the formula for translational kinetic energy, can be a valuable problem-solving tool.
The kinetic energy of a rotating rigid body is:

$$KE = \frac{1}{2}I\omega^2$$

Considering that I is the rotational equivalent for mass and ω is the rotational equivalent for velocity, this equation should come as no surprise.

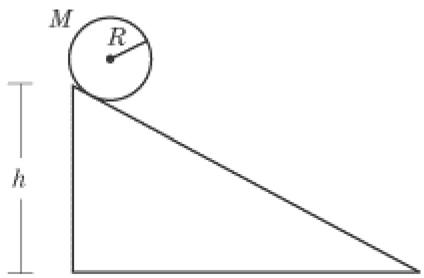
An object, such as a pool ball, that is spinning as it travels through space, will have both rotational and translational kinetic energy:

$$KE = \frac{1}{2}Mv_{\mathrm{cm}} + \frac{1}{2}I\omega^2$$

In this formula, M is the total mass of the rigid body and v_{cm} is the velocity of its center of mass.

This equation comes up most frequently in problems involving a rigid body that is rolling along a surface without sliding. Unlike a body sliding along a surface, there is no kinetic friction to slow the body's motion. Rather, there is static friction as each point of the rolling body makes contact with the surface, but this static friction does not work on the rolling object and dissipates no energy.

EXAMPLE



A wheel of mass *M* and radius *R* is released from rest and rolls to the bottom of an inclined plane of height *h* without slipping. What is its velocity at the bottom of the incline? The moment of inertia of a wheel of mass *M* and radius *R* rotating about an axis through its center of mass is $\frac{1}{2} MR^2$.

Because the wheel loses no energy to friction, we can apply the law of conservation of mechanical energy. The change in the wheel's potential energy is $-mgh$. The change in the wheel's kinetic energy is $\frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I\omega^2$. Applying conservation of mechanical energy:

$$-Mgh + \frac{1}{2}Mv^2 + \frac{1}{2}(\frac{1}{2}MR^2)(\frac{v^2}{R^2}) = 0$$

$$Mgh = (\frac{1}{2} + \frac{1}{4})Mv^2$$

$$v = \sqrt{\frac{4}{3}gh}$$

It's worth remembering that an object rolling down an incline will pick up speed more slowly than an object sliding down a frictionless incline. Rolling objects pick up speed more slowly because only some of the kinetic energy they gain is converted into translational motion, while the rest is converted into rotational motion.

Angular Momentum

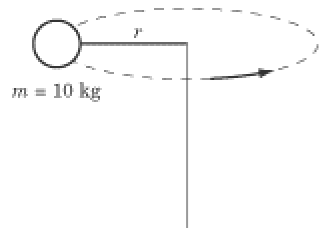
The rotational analogue of linear momentum is **angular momentum**, L. After torque and equilibrium, angular momentum is the aspect of rotational motion most likely to be tested on SAT II Physics. For the test, you will probably have to deal only with the angular momentum of a particle or body moving in a circular trajectory. In such a case, we can define angular momentum in terms of moment of inertia and angular velocity, just as we can define linear momentum in terms of mass and velocity:

$$L = I\omega$$

The angular momentum vector always points in the same direction as the angular velocity vector.

Angular Momentum of a Single Particle

Let's take the example of a tetherball of mass m swinging about on a rope of length r:



The tetherball has a moment of inertia of $I = mr^2$ and an angular velocity of $\omega = v/r$. Substituting these values into the formula for linear momentum we get:

$$L = I\omega = (mr^2)(v/r)$$
$$= mvr$$

This is the value we would expect from the cross product definition we saw earlier of angular momentum. The momentum, $\mathbf{p} = m\mathbf{v}$ of a particle moving in a circle is always tangent to the circle and perpendicular to the radius. Therefore, when a particle is moving in a circle,

$$\mathbf{L} = pr \sin 90° = pr = mvr$$

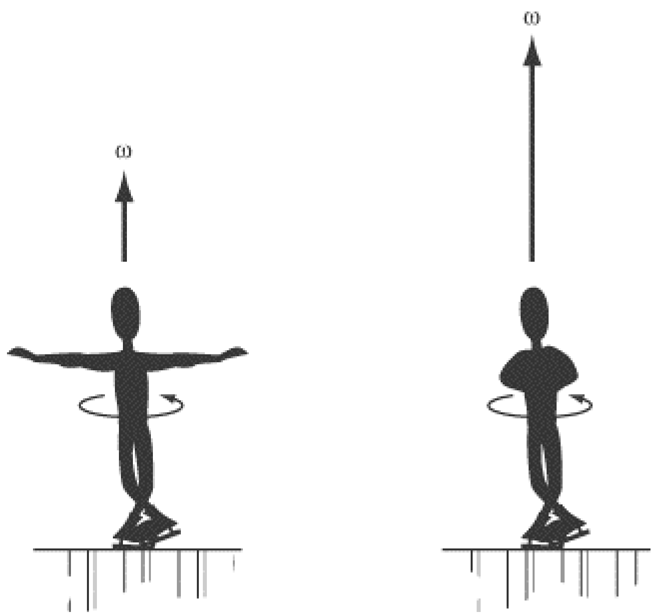
Newton's Second Law and Conservation of Angular Momentum

In the previous chapter, we saw that the net force acting on an object is equal to the rate of change of the object's momentum with time. Similarly, the net torque acting on an object is equal to the rate of change of the object's angular momentum with time:

$$\boldsymbol{\tau}_{\text{net}} = \frac{\Delta \mathbf{L}}{\Delta t}$$

If the net torque action on a rigid body is zero, then the angular momentum of the body is constant or conserved. The **law of conservation of angular momentum** is another one of nature's beautiful properties, as well as a very useful means of solving problems. It is likely that angular momentum will be tested in a conceptual manner on SAT II Physics.

EXAMPLE



One of Brian Boitano's crowd-pleasing skating moves involves initiating a spin with his arms extended and then moving his arms closer to his body. As he does so, he spins at a faster and faster rate. Which of the following laws best explains this phenomenon?

(A) Conservation of Mechanical Energy

(B) Conservation of Angular Momentum

(C) Conservation of Linear Momentum

(D) Newton's First Law

(E) Newton's Second Law

Given the context, the answer to this question is no secret: it's **B**, the conservation of angular momentum. Explaining why is the interesting part.

As Brian spins on the ice, the net torque acting on him is zero, so angular momentum is conserved. That means that $I\omega$ is a conserved quantity. I is proportional to R^2, the distance of the parts of Brian's body from his axis of rotation. As he draws his arms in toward his body, his mass is more closely concentrated about his axis of rotation, so I decreases. Because $I\omega$ must remain constant, ω must increase as I decreases. As a result, Brian's angular velocity increases as he draws his arms in toward his body.

Key Formulas

Angular Position	$\phi = \frac{l}{r}$

Definition of a Radian	$1 \text{ revolution} = 2\pi \text{rad} = 360°$
Average Angular Velocity	$\bar{\omega} = \frac{\Delta\theta}{\Delta t}$
Average Angular Acceleration	$\bar{\alpha} = \frac{\Delta\omega}{\Delta t}$
Angular Frequency	$f = \frac{\omega}{2\pi}$
Angular Period	$T = \frac{1}{f}$
Relations between Linear and Angular Variables	$s = \theta r$ $v = \omega r$ $a = \alpha r$
Equations for Rotational and Angular Kinematics with Constant Acceleration	$\phi = \phi_0 + \frac{1}{2}(\omega + \omega_0)t$ $\omega = \omega_0 + \alpha t$ $\phi = \phi_0 + \omega_0 t + \frac{1}{2}\alpha t^2$ $\phi = \phi_0 + \omega t - \frac{1}{2}\alpha t^2$ $\omega^2 = \omega_0^2 + 2\alpha(\phi - \phi_0)$
Torque As Trigonometric Function	$\tau = Fr \sin\theta$
Component Form of the Torque Equation	$\tau = \boldsymbol{F}_\perp \boldsymbol{r} = \boldsymbol{F}\boldsymbol{r}_\perp$
Torque As Cross Product	$\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F}$
Newton's Second Law in Terms of Rotational Motion	$\boldsymbol{\tau}_{\text{net}} = I\boldsymbol{\alpha}$
Moment of Inertia	$I = \Sigma mr^2$
Kinetic Energy of Rotation	$KE = \frac{1}{2}I\omega^2$
Angular Momentum of a Particle	$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$

Component Form of the Angular Momentum of a Particle	$L = mrv_{\perp}$ $L = mr_{\perp}v$
Angular Momentum of a Rotating Rigid Body	$L = I\omega$

Practice Questions

1. The instantaneous velocity of a point on the outer edge of a disk with a diameter of 4 m that is rotating at 120 revolutions per minute is most nearly:

   (A) 4 m/s

   (B) 6 m/s

   (C) 12 m/s

   (D) 25 m/s

   (E) 50 m/s

2. A washing machine, starting from rest, accelerates within 3.14 s to a point where it is revolving at a frequency of 2.00 Hz. Its angular acceleration is most nearly:

   (A) 0.100 rad/s^2

   (B) 0.637 rad/s^2

   (C) 2.00 rad/s^2

   (D) 4.00 rad/s^2

   (E) 6.28 rad/s^2

3. What is the direction of the angular velocity vector for the second hand of a clock going from 0 to 30 seconds?

   (A) Outward from the clock face

   (B) Inward toward the clock face

   (C) Upward

   (D) Downward

   (E) To the right

4. Which of the following are means of maximizing the torque of a force applied to a rotating object?

   I. Maximize the magnitude of the applied force
   II. Apply the force as close as possible to the axis of rotation
   III. Apply the force perpendicular to the displacement vector between the axis of rotation and the point of applied force

   (A) I only

   (B) II only

   (C) I and II only

   (D) I and III only

   (E) I, II, and III

5. What is the torque on the pivot of a pendulum of length *R* and mass *m*, when the mass is at an

angle θ?

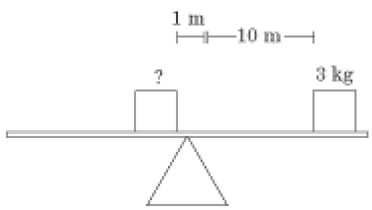
(A) $m\frac{g}{R}\sin\theta$

(B) $m\frac{g}{R}\cos\theta$

(C) $mgR\sin\theta$

(D) $mgR\cos\theta$

(E) $mgR\tan\theta$



6. Two objects rest on a seesaw. The first object has a mass of 3 kg and rests 10 m from the pivot. The other rests 1 m from the pivot. What is the mass of the second object if the seesaw is in equilibrium?

(A) 0.3 kg

(B) 3 kg

(C) 10 kg

(D) 30 kg

(E) 50 kg

7. What is the angular acceleration of a 0.1 kg record with a radius of 0.1 m to which a torque of 0.05 N Â· m is applied? The moment of inertia of a disk spinning about its center is $^1/_2MR^2$.

(A) 0.1 rad/s^2

(B) 0.5 rad/s^2

(C) 1 rad/s^2

(D) 5 rad/s^2

(E) 10 rad/s^2

8. A disk of mass *m* and radius *R* rolls down an inclined plane of height *h* without slipping. What is the velocity of the disk at the bottom of the incline? The moment of inertia for a disk is $^1/_2mR^2$.

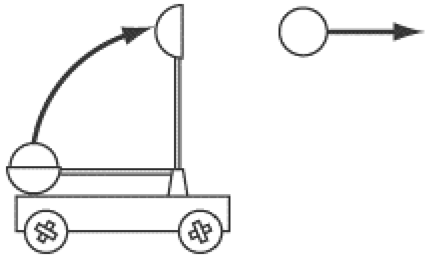
(A) $\sqrt{gh}$

(B) $\sqrt{\frac{4}{3}gh}$

(C) $\sqrt{2gh}$

(D) $2\sqrt{gh}$

(E) $2\sqrt{2gh}$



9. A catapult with a basket of mass 50 kg launches a 200 kg rock by swinging around from a horizontal to a vertical position with an angular velocity of 2.0 rad/s. Assuming the rest of the catapult is mass less and the catapult arm is 10 m long, what is the velocity of the rock as it leaves the catapult?

   (A) 10 m/s

   (B) 20 m/s

   (C) 25 m/s

   (D) 50 m/s

   (E) 100 m/s

10. How should the mass of a rotating body of radius *r* be distributed so as to maximize its angular velocity?

   (A) The mass should be concentrated at the outer edge of the body

   (B) The mass should be evenly distributed throughout the body

   (C) The mass should be concentrated at the axis of rotation

   (D) The mass should be concentrated at a point midway between the axis of rotation and the outer edge of the body

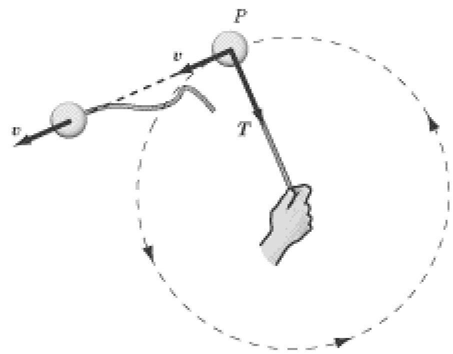
   (E) Mass distribution has no impact on angular velocity

Circular Motion and Gravitation

NEWTON'S FIRST LAW TELLS US THAT objects will move in a straight line at a constant speed unless a net force is acting upon them. That rule would suggest that objects moving in a circle—whether they're tetherballs or planets—are under the constant influence of a changing force, since their trajectory is not in a straight line. We will begin by looking at the general features of circular motion and then move on to examine gravity, one of the principal sources of circular motion.

Uniform Circular Motion

Uniform circular motion occurs when a body moves in a circular path with constant speed. For example, say you swing a tethered ball overhead in a circle:

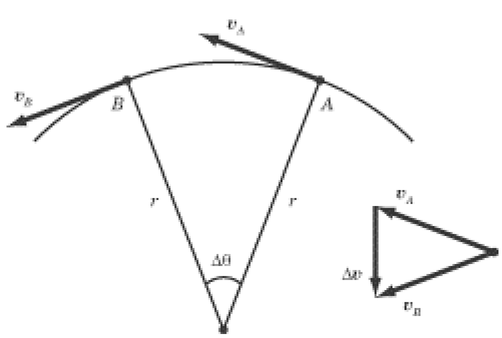


If we leave aside gravity for the moment, the only force acting on the ball is the force of tension, $\boldsymbol{T}$, of the string. This force is always directed radially inward along the string, toward your hand. In other words, the force acting on a tetherball traveling in a circular path is always directed toward the center of that circle.

Note that although the direction of the ball's velocity changes, the ball's velocity is constant in magnitude and is always tangent to the circle.

Centripetal Acceleration

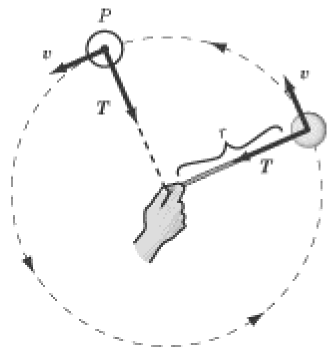
From kinematics, we know that acceleration is the rate of change of the velocity vector with time. If we consider two points very close together on the ball's trajectory and calculate $\Delta \boldsymbol{v}$, we find that the ball's acceleration points inward along the radius of the circle.



The acceleration of a body experiencing uniform circular motion is always directed toward the center of the circle, so we call that acceleration **centripetal acceleration**, a_c *Centripetal* comes from a Latin word meaning "center-seeking." We define the centripetal acceleration of a body moving in a circle as:

$$a_c = \frac{v^2}{r}$$

where *v* is the body's velocity, and *r* is the radius of the circle. The body's centripetal acceleration is constant in magnitude but changes in direction. Note that even though the direction of the centripetal acceleration vector is changing, the vector always points toward the center of the circle.



How This Knowledge Will Be Tested

Most of us are accustomed to think of "change" as a change in magnitude, so it may be counterintuitive to think of the acceleration vector as "changing" when its magnitude remains constant. You'll frequently find questions on SAT II Physics that will try to catch you sleeping on the nature of centripetal acceleration. These questions are generally qualitative, so if you bear in mind that the acceleration vector is constant in magnitude, has a direction that always points toward the center of the circle, and is always perpendicular to the velocity vector, you should have no problem at all.

Centripetal Force

Wherever you find acceleration, you will also find force. For a body to experience centripetal acceleration, a **centripetal force** must be applied to it. The vector for this force is similar to the acceleration vector: it is of constant magnitude, and always points radially inward to the center of the circle, perpendicular to the velocity vector.

We can use Newton's Second Law and the equation for centripetal acceleration to write an equation for the centripetal force that maintains an object's circular motion:

$$F = ma_c = \frac{mv^2}{r}$$

EXAMPLE

> A ball with a mass of 2 kg is swung in a circular path on a mass less rope of length 0.5 m. If the ball's speed is 1 m/s, what is the tension in the rope?

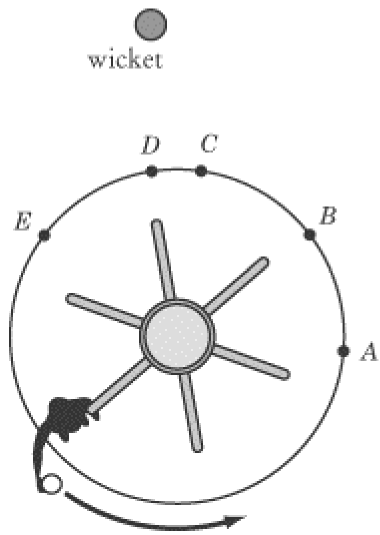
The tension in the rope is what provides the centripetal force, so we just need to calculate the centripetal force using the equation above:

$$T = \frac{mv^2}{r} = \frac{(2\ \text{kg})(1\ \text{m/s})^2}{0.5\ \text{m}}$$
$$= 4\ \text{N}$$

Objects Released from Circular Motion

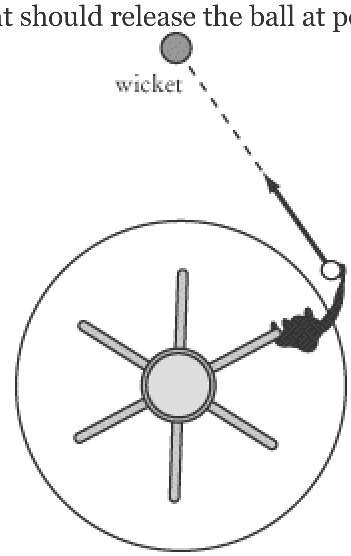
One concept that is tested frequently on SAT II Physics is the trajectory of a circling body when the force providing centripetal acceleration suddenly vanishes. For example, imagine swinging a ball in a circle overhead and then letting it go. As soon as you let go, there is no longer a centripetal force acting on the ball. Recall Newton's First Law: when no net force is acting on an object, it will move with a constant velocity. When you let go of the ball, it will travel in a straight line with the velocity it had when you let go of it.

EXAMPLE



A student is standing on a merry-go-round that is rotating counterclockwise, as illustrated above. The student is given a ball and told to release it in such a way that it knocks over the wicket at the top of the diagram. At what point should the student release the ball?

When the student releases the ball, it will travel in a straight line, tangent to the circle. In order to hit the wicket, then, the student should release the ball at point *B*.



Newton's Law of Universal Gravitation

Newton's Law of Universal Gravitation is a fundamental physical law. We experience its effects everywhere on this planet, and it is the prime mover in the vast world of astronomy. It can also be expressed in a relatively simple mathematical formula on which SAT II Physics is almost certain to test you.

Gravitational Force

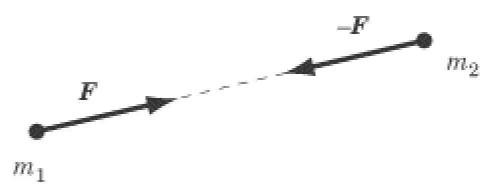
In 1687, Isaac Newton published his Law of Gravitation in *Philosophiae Naturalis Principia Mathematica*. Newton proposed that everybody in the universe is attracted to every other body with a force that is directly proportional to the product of the bodies' masses and inversely proportional to the square of the bodies' separation. In terms of mathematical relationships, Newton's Law of Gravitation states that the force of gravity, F_g, between two particles of mass m_1 and m_2 has a magnitude of:

$$F_g = G\frac{m_1 m_2}{r^2}$$

where r is the distance between the center of the two masses and G is the **gravitational constant**. The value of G was determined experimentally by Henry Cavendish in 1798:

$$G = 6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$$

The force of gravity is a vector quantity. Particle m_1 attracts particle m_2 with a force that is directed toward m_1, as illustrated in the figure below. Similarly, particle m_2 attracts particle m_1 with a force that is directed toward m_2.



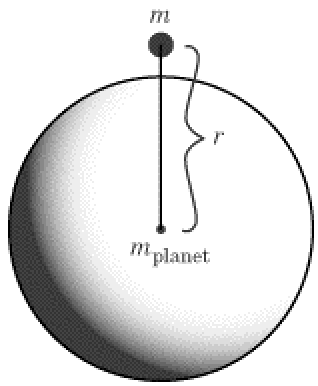
Note that the gravitational force, F_g, acting on particle m_1 is equal and opposite to the gravitational force acting on particle m_2, $-F_g$. This is a consequence of Newton's Third Law.

Let's consider two examples to give you a more intuitive feel for the strength of the gravitational force. The force of gravity between two oranges on opposite sides of a table is quite tiny, roughly 10^{-13} N. On the other hand, the gravitational force between two galaxies separated by 10^6 light years is something in the neighborhood of 10^{27} N!

Newton's Law of Gravitation was an enormous achievement, precisely because it synthesized the laws that govern motion on Earth and in the heavens. Additionally, Newton's work had a profound effect on philosophical thought. His research implied that the universe was a rational place that could be described by universal, scientific laws. But this is knowledge for another course. If you are interested in learning more about it, make sure to take a class on the history of science in college.

Gravity on the Surface of Planets

Previously, we noted that the acceleration due to gravity on Earth is 9.8 m/s² toward the center of the Earth. We can derive this result using Newton's Law of Gravitation.



Consider the general case of a mass accelerating toward the center of a planet. Applying Newton's Second Law, we find:

$$F = m_{\text{object}} a = G\frac{m_{\text{object}} m_{\text{planet}}}{r^2}$$
$$a = G\frac{m_{\text{planet}}}{r^2}$$

Note that this equation tells us that acceleration is directly proportional to the mass of the planet and inversely proportional to the square of the radius. The mass of the object under the influence of the planet's gravitational pull doesn't factor into the equation. This is now pretty common knowledge, but it still trips up students on SAT II Physics: all objects under the influence of gravity, regardless of mass, fall with the same acceleration.

Acceleration on the Surface of the Earth

To find the acceleration due to gravity on the surface of the Earth, we must substitute values for the gravitational constant, the mass of the Earth, and the radius of the Earth into the equation above:

$$a = (6.67 \times 10^{-11}\ \text{N} \cdot \text{m}^2/\text{kg}^2)\frac{5.98 \times 10^{24}\ \text{kg}}{(6.37 \times 10^6\ \text{m})^2}$$
$$= 9.8\ \text{m/s}^2$$

Not coincidentally, this is the same number we've been using in all those kinematic equations.

Acceleration Beneath the Surface of the Earth

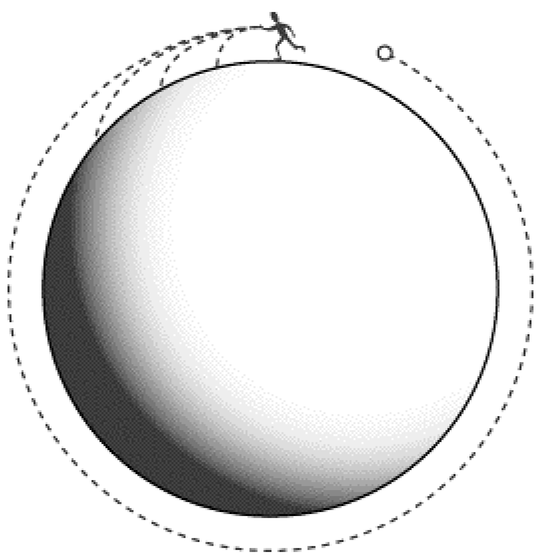
If you were to burrow deep into the bowels of the Earth, the acceleration due to gravity would be different. This difference would be due not only to the fact that the value of r would have decreased. It would also be due to the fact that not all of the Earth's mass would be under you. The mass above your head wouldn't draw you toward the center of the Earth—quite the opposite—and so the value of m_{planet} would also decrease as you burrowed. It turns out that there is a linear relationship between the acceleration due to gravity and one's distance from the Earth's center when you are beneath the surface of the Earth. Burrow halfway to the center of the Earth and the acceleration due to gravity will be $^1/_2\, g$. Burrow three-quarters of the way to the center of the Earth and the acceleration due to gravity will be $^1/_4\, g$.

Orbits

The **orbit** of satellites—whether of artificial satellites or natural ones like moons and planets—is a common way in which SAT II Physics will test your knowledge of both uniform circular motion and gravitation in a single question.

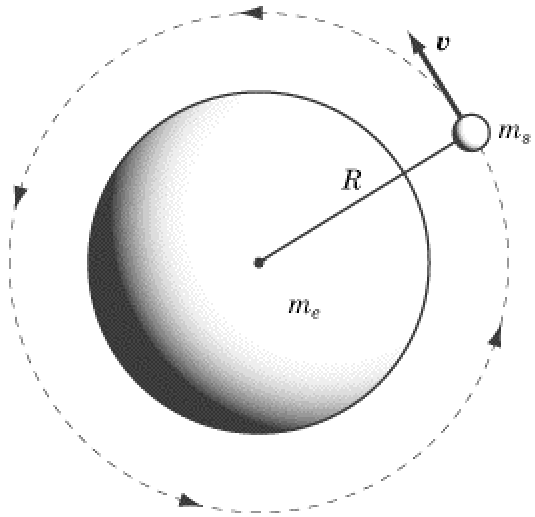
How Do Orbits Work?

Imagine a baseball pitcher with a very strong arm. If he just tosses the ball lightly, it will fall to the ground right in front of him. If he pitches the ball at 100 miles per hour in a line horizontal with the Earth, it will fly somewhere in the neighborhood of 80 feet before it hits the ground. By the same token, if he were to pitch the ball at 100,000miles per hour in a line horizontal with the Earth, it will fly somewhere in the neighborhood of 16 miles before it hits the ground. Now remember: the Earth is round, so if the ball flies far enough, the ball's downward trajectory will simply follow the curvature of the Earth until it makes a full circle of the Earth and hits the pitcher in the back of the head. A satellite in orbit is an object in free fall moving at a high enough velocity that it falls around the Earth rather than back down to the Earth.



Gravitational Force and Velocity of an Orbiting Satellite

Let's take the example of a satellite of mass m_s orbiting the Earth with a velocity v. The satellite is a distance R from the center of the Earth, and the Earth has a mass of m_e.



The centripetal force acting on the satellite is the gravitational force of the Earth. Equating the formulas for gravitational force and centripetal force we can solve for v:

$$G\frac{m_s m_e}{R^2} = \frac{m_s v^2}{R}$$
$$v = \sqrt{\frac{G m_e}{R}}$$

As you can see, for a planet of a given mass, each radius of orbit corresponds with a certain velocity. That is, any object orbiting at radius R must be orbiting with a velocity of $\sqrt{Gm_e/R}$. If the satellite's speed is too slow, then the satellite will fall back down to Earth. If the satellite's speed is too fast, then the satellite will fly out into space.

Gravitational Potential Energy

In Chapter 4, we learned that the potential energy of a system is equal to the amount of work that must be done to arrange the system in that particular configuration. We also saw that **gravitational potential energy** depends on how high an object is off the ground: the higher an object is, the more work needs to be done to get it there.

Gravitational potential energy is not an absolute measure. It tells us the amount of work needed to move an object from some arbitrarily chosen reference point to the position it is presently in. For instance, when dealing with bodies near the surface of the Earth, we choose the ground as our reference point, because it makes our calculations easier. If the ground is $h = 0$, then for a height h above the ground an object has a potential energy of mgh.

Gravitational Potential in Outer Space

Off the surface of the Earth, there's no obvious reference point from which to measure gravitational potential energy. Conventionally, we say that an object that is an infinite distance away from the Earth has zero gravitational potential energy with respect to the Earth. Because a negative amount of work is done to bring an object closer to the Earth, gravitational potential energy is always a negative number when using this reference point.

The gravitational potential energy of two masses, m_1 and m_2, separated by a distance r is:

$$U = -G\frac{m_1 m_2}{r}$$

EXAMPLE

> A satellite of mass m_s is launched from the surface of the Earth into an orbit of radius $2r_e$, where r_e is the radius of the Earth. How much work is done to get it into orbit?

The work done getting the satellite from one place to another is equal to the change in the satellite's potential energy. If its potential energy on the surface of the Earth is U_1 and its potential energy when it is in orbit is U_2, then the amount of work done is:

$$\begin{aligned} W = U_2 - U_1 &= -G\frac{m_s m_e}{2r_e} - -G\frac{m_s m_e}{r_e} \\ &= G\frac{m_s m_e}{r_e} - G\frac{m_s m_e}{2r_e} \\ &= G\frac{m_s m_e}{2r_e} \end{aligned}$$

Energy of an Orbiting Satellite

Suppose a satellite of mass m_s is in orbit around the Earth at a radius R. We know the kinetic energy of the satellite is $KE = \frac{1}{2} mv^2$. We also know that we can express centripetal force, F_c, as $F_c = mv^2/R$. Accordingly, we can substitute this equation into the equation for kinetic energy and get:

$$KE = \frac{1}{2}F_c R$$

Because F_c is equal to the gravitational force, we can substitute Newton's Law of Universal Gravitation in for F_c:

$$KE = G\frac{m_s m_e}{2R}$$

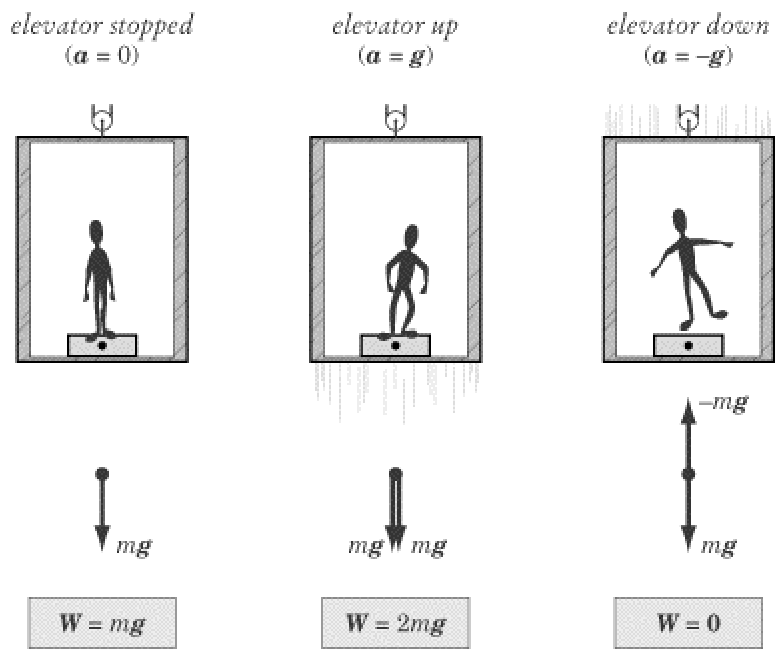
We know that the potential energy of the satellite is $U = -Gm_s m_e/R$, so the total energy of the satellite is the sum, $E = KE + U$:

$$E = -G\frac{m_s m_e}{R} + G\frac{m_s m_e}{2R} = -G\frac{m_s m_e}{2R}$$

Weightlessness

People rarely get to experience firsthand the phenomenon of **weightlessness**, but that doesn't keep SAT II Physics from testing you on it. There is a popular misconception that astronauts in satellites experience weightlessness because they are beyond the reach of the Earth's gravitational pull. If you already know this isn't the case, you're in a good position to answer correctly anything SAT II Physics may ask about weightlessness.

In order to understand how weightlessness works, let's look at the familiar experience of gaining and losing weight in an elevator. Suppose you bring a bathroom scale into the elevator with you to measure your weight.



When the elevator is at rest, the scale will read your usual weight, $W = mg$, where m is your mass. When the elevator rises with an acceleration of g, you will be distressed to read that your weight is now $m(g + g) = 2mg$. If the elevator cable is cut so that the elevator falls freely with an acceleration of $-g$, then your weight will be $m(g - g) = 0$.

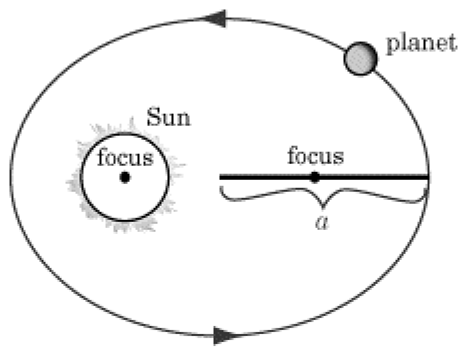
While in free fall in the elevator, if you were to take a pen out of your pocket and "drop" it, it would just hover in the air next to you. You, the pen, and the elevator are all falling at the same rate, so you are all motionless relative to one another. When objects are in free fall, we say that they experience weightlessness. You've probably seen images of astronauts floating about in space shuttles. This is not because they are free from the Earth's gravitational pull. Rather, their space shuttle is in orbit about the Earth, meaning that it is in a perpetual free fall. Because they are in free fall, the astronauts, like you in your falling elevator, experience weightlessness.

Weightless environments provide an interesting context for testing Newton's Laws. Newton's First Law tells us that objects maintain a constant velocity in the absence of a net force, but we're so used to being in an environment with gravity and friction that we never really see this law working to its full effect. Astronauts, on the other hand, have ample opportunity to play around with the First Law. For example, say that a weightless astronaut is eating lunch as he orbits the Earth in the space station. If the astronaut releases his grasp on a tasty dehydrated strawberry, then the berry, like your pen, floats in midair exactly where it was "dropped." The force of gravity exerted by the Earth on the strawberry causes the strawberry to move in the same path as the spaceship. There is no relative motion between the astronaut and the berry unless the astronaut, or something else in the spaceship, exerts a net force on the berry.

Kepler’s Laws

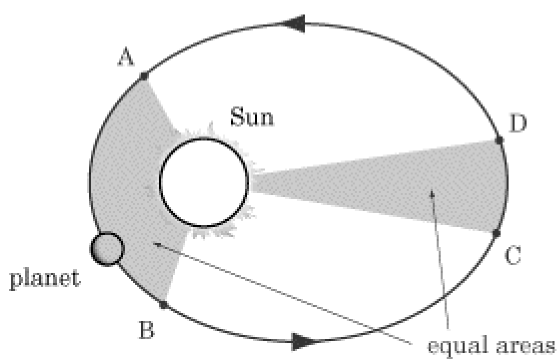
After poring over the astronomical observations of his mentor Tycho Brahe (1546–1601), Johannes Kepler (1571–1630) determined three laws of planetary motion. These laws are of great significance, because they formed the background to Newton’s thinking about planetary interaction and the attraction between masses. In fact, Newton later showed that Kepler’s Laws could be derived mathematically from his own Law of Universal Gravitation and laws of motion, providing evidence in favor of Newton’s new theories. Another point in favor of their significance is that any one of them may appear on SAT II Physics.

Kepler’s First Law states that the path of each planet around the sun is an ellipse with the sun at one focus.



a = semimajor axis

Kepler’s Second Law relates a planet’s speed to its distance from the sun. Because the planets’ orbits are elliptical, the distance from the sun varies. The Second Law states that if a line is drawn from the sun to the orbiting planet, then the area swept out by this line in a given time interval is constant. This means that when the planet is farthest from the sun it moves much more slowly than when it is closest to the sun.



It is important to remember that although Kepler formulated this law in reference to planets moving around the sun, it also holds true for astronomical objects, like comets, that also travel in elliptical orbits around the sun.

Kepler's Third Law states that given the period, T, and semimajor axis, a, of a planet's elliptical orbit, the ratio T^2/a^3 is the same for every planet. The semimajor axis is the longer one, along which the two foci are located.

EXAMPLE

> Every 76 years, Halley's comet passes quite close by the Earth. At the most distant point in its orbit, it is much farther from the sun even than Pluto. Is the comet moving faster when it is closer to Earth or closer to Pluto?

According to Kepler's Second Law, objects that are closer to the sun orbit faster than objects that are far away. Therefore, Halley's comet must be traveling much faster when it is near the Earth than when it is off near Pluto.

Key Formulas

Centripetal Acceleration	$a_c = \frac{v^2}{r}$
Centripetal Force	$F = \frac{mv^2}{r}$
Newton's Law of Universal Gravitation	$F_g = G\frac{m_1 m_2}{r^2}$
Acceleration Due to Gravity at the Surface of a Planet	$a = G\frac{m_{\text{planet}}}{r^2}$
Velocity of a Satellite in Orbit	$v = \sqrt{\frac{Gm_e}{R}}$
Gravitational Potential Energy	$U = -G\frac{m_1 m_2}{r}$
Kinetic Energy of a Satellite in Orbit	$KE = G\frac{m_s m_e}{2R}$
Total Energy of a Satellite in Orbit	$E = -G\frac{m_s m_e}{2R}$
Kepler's Third Law	$\frac{T^2}{a^3} = \text{constant}$

Practice Questions

Questions 1–3 refer to a ball of mass m on a string of length R, swinging around in circular motion, with instantaneous velocity v and centripetal acceleration a.

1. What is the centripetal acceleration of the ball if the length of the string is doubled?

    (A) $a/4$

    (B) $a/2$

    (C) a

    (D) $2a$

    (E) $4a$

2. What is the centripetal acceleration of the ball if the instantaneous velocity of the ball is doubled?

    (A) $a/4$

    (B) $a/2$

    (C) a

    (D) $2a$

    (E) $4a$

3. What is the centripetal acceleration of the ball if its mass is doubled?

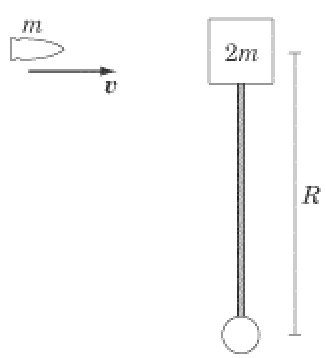
    (A) $a/4$

    (B) $a/2$

    (C) a

    (D) $2a$

    (E) $4a$



4. A bullet of mass m traveling at velocity v strikes a block of mass $2m$ that is attached to a rod of length R. The bullet collides with the block at a right angle and gets stuck in the block. The rod is free to rotate. What is the centripetal acceleration of the block after the collision?

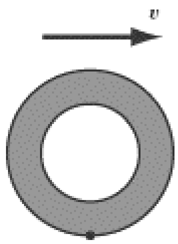
    (A) v^2/R

    (B) $(1/2)v^2/R$

    (C) $(1/3)v^2/R$

    (D) $(1/4)v^2/R$

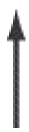
    (E) $(1/9)v^2/R$



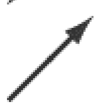
5. A car wheel drives over a pebble, which then sticks to the wheel momentarily as the wheel displaces it. What is the direction of the initial acceleration of the pebble?

(A) 

(B) 

(C) 

(D) 

(E) 

6. If we consider the gravitational force *F* between two objects of masses m_1 and m_2 respectively, separated by a distance *R*, and we double the distance between them, what is the new magnitude of the gravitational force between them?

(A) $F/4$

(B) $F/2$

(C) F

(D) $2F$

(E) $4F$

7. If the Earth were compressed in such a way that its mass remained the same, but the distance around the equator were just one-half what it is now, what would be the acceleration due to gravity at the surface of the Earth?

(A) $g/4$

(B) $g/2$

(C) g

(D) $2g$

(E) $4g$

8. A satellite orbits the Earth at a radius *r* and a velocity *v*. If the radius of its orbit is doubled, what is its velocity?

(A) $v/2$

(B) $v/\sqrt{2}$

(C) v

(D) $\sqrt{2}\,v$

(E) $2v$

9. An object is released from rest at a distance of $2r_e$ from the center of the Earth, where r_e is the radius of the Earth. In terms of the gravitational constant (G), the mass of the Earth (M), and r_e, what is the velocity of the object when it hits the Earth?

(A) $\sqrt{GM/r_e}$

(B) GM/r_e

(C) $\sqrt{GM/2r_e}$

(D) $GM/2r_e$

(E) $2GM/r_e$

10. Two planets, A and B, orbit a star. Planet A moves in an elliptical orbit whose semimajor axis has length a. Planet B moves in an elliptical orbit whose semimajor axis has a length of $9a$. If planet A orbits with a period T, what is the period of planet B's orbit?

(A) $729T$

(B) $27T$

(C) $3T$

(D) $T/3$

(E) $T/27$